

D=6 , N=1 String Vacua and Duality<sup>1</sup>

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**Abstract**

We review the structure  $D = 6$ ,  $N = 1$  string vacua with emphasis on the different connections due to  $T$ -dualities and  $S$ -dualities. The topics discussed include: Anomaly cancellation; K3 and orbifold  $D = 6$ ,  $N = 1$  heterotic compactifications;  $T$ -dualities between  $E_8 \times E_8$  and  $Spin(32)/Z_2$  heterotic vacua; non-perturbative heterotic vacua and small instantons;  $N = 2$  Type-II/Heterotic duality in  $D = 4$ ; F-theory/heterotic duality in  $D = 6$ ; and heterotic/heterotic duality in six and four dimensions.

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# 1 Introduction

The last few years of work on non-perturbative  $S$ -dualities [1] have taught us a lot about the connections and equivalences among different string vacua in different dimensions (for reviews, see ref.[2]). Also a good amount of non-perturbative aspects of string theories has been learnt. Of course, one would be particularly interested in the understanding of non-perturbative vacua in  $D = 4$  with  $N = 1$  or  $N = 0$  supersymmetries. However, as one goes to lower dimensions and smaller number of supersymmetries, the physics becomes more and more non-trivial. It is thus interesting to proceed step by step and try first to understand as much as possible of the dynamics of theories with higher number of dimensions and/or supersymmetries. In this respect, six dimensional vacua with the minimal number of supersymmetries,  $N = 1$ , are particularly interesting because of several reasons: 1) These theories are chiral and the cancellation of anomalies restricts the physics substantially. One can use this constraint both as a check of the consistency of the vacua and as a guide for the search of new non-perturbative dynamics. 2) When toroidally compactified, these theories yield 4-dimensional vacua with  $N = 2$  supersymmetry, for which a number of non-perturbative results are known. 3) One expects that some of the non-perturbative physics going on for  $D = 6$ ,  $N = 1$  theories have a reflection in  $D = 4$ . Thus, for example, one can consider 4-dimensional  $N = 1$  vacua obtained upon heterotic compactification on a CY manifold which is a K3 fibration. When the size of the base is large the theory looks locally like the heterotic compactified on K3 and some non-perturbative phenomena (like e.g. small instanton effects) are inherited from known  $D = 6$  dynamics; 4) Some  $D = 6$  string vacua have suggested the existence of new classes of non-trivial renormalization group fixed points of  $D = 6$  field theory [3, 4, 5, 6]. At special points in the moduli space of some of the  $D = 6$ ,  $N = 1$  vacua non-critical tensionless strings appear, associated to these new classes of non-trivial field theories.

$D = 6$ ,  $N = 1$  vacua have been constructed in essentially three ways: 1) heterotic compactifications on a K3 manifold (or orbifold); 2) F-theory compactified on elliptic Calabi-Yau threefolds ; 3) Type-IIB  $Z_N$  orientifolds. All these constructions are related and one can often construct the same model (possibly in different regions of the moduli

space) by using different techniques. In these lectures we will mostly discuss theories constructed using the first two. The structure of these lectures is as follows. In chapter 2, after reviewing the constraints coming from anomaly cancellation, we describe the construction of  $D = 6$ ,  $N = 1$  heterotic vacua in terms of toroidal orbifolds and K3 compactifications. The  $T$ -dualities among  $E_8 \times E_8$  and  $Spin(32)/Z_2$  vacua are also discussed. In section 2.3 non-perturbative heterotic vacua and their connection with the physics of small instantons are summarized. In chapter 3 we first discuss the structure of  $D = 4$ ,  $N = 2$  string vacua. They are of interest for the purposes of these lectures since these vacua appear e.g. after trivial toroidal compactification of  $D = 6$ ,  $N = 1$  theories. We then describe the dualities between Type-IIA theory compactified on a Calabi-Yau and heterotic compactified on  $K3 \times T^2$ . This is better understood from the perspective of the  $D = 6$  duality between F-theory and heterotic string [7, 8, 9] which is discussed in some detail in the rest of the chapter. In the last chapter we discuss heterotic/heterotic duality [10, 11, 12, 13, 14, 15] from different perspectives including usual heterotic K3 compactifications, Type-IIB orientifolds and F-theory.

## 2 $D = 6$ , $N = 1$ Heterotic Vacua

### 2.1 Gauge and Gravitational Anomalies

The relevant supermultiplets in a  $D = 6$ ,  $N = 1$  theory are as follows:

$$\begin{aligned}
\text{Gravity} &\rightarrow (3, 3) + 2(2, 3) + (1, 3) \\
\text{Tensor} &\rightarrow (3, 1) + 2(2, 1) + (1, 1) \\
\text{Vector} &\rightarrow (2, 2) + 2(1, 2) \\
\text{Hypermultiplet} &\rightarrow 2(2, 1) + 4(1, 1)
\end{aligned} \tag{2.1}$$

where the transformation properties with respect to the little group  $Spin(4) \simeq SU(2) \times SU(2)$  group are shown. Notice the following relevant facts: 1) There are scalar fields only in the hypermultiplets (two complex scalars) and in the tensor multiplets (one real scalar). The vector multiplet does not contain scalars so there is no Coulomb phase associated to the vector multiplets in  $D = 6$ . On the other hand there is a Coulomb

phase associated to tensor multiplets [4] since they contain a scalar and, upon reduction to lower dimensions, a vector boson appears from the two index antisymmetric field.

2) The gravity multiplet contains a self-dual two index antisymmetric field  $B_{\mu\nu}^+$  and the tensor multiplets contain anti-selfdual two-form fields  $B_{\mu\nu}^-$ . In a theory with just one tensor multiplet the  $B_{\mu\nu}^-$  can combine with the  $B_{\mu\nu}^+$  from the gravity multiplet to form an unconstrained field, for which a local action may be written. This is the case of perturbative compactifications of the heterotic string down to six dimensions for which a single tensor multiplet is inherited from the  $D = 10$  gravity multiplet. It is also the case for *smooth* K3 compactifications of Type-I theory. There is no known local Lagrangian description for theories with more than one tensor multiplet although, as we will see later on, theories of that type frequently appear in heterotic non-perturbative string vacua. In Type-I theory, tensor multiplets appear even at the perturbative level in the presence of orbifold singularities in the compact manifold [16, 17, 18].

3) Due to supersymmetry, the scalars in hypermultiplets and those in tensor multiplets are decoupled. In particular the metric in the hypermultiplet moduli space is independent of the tensors and viceversa. Also, the kinetic terms of vector multiplets only depend on the tensor multiplets and not on the hypermultiplets [19].

4) At the perturbative level, the only dynamics present in this class of theories is that of the Higgs mechanism. In this process the number of hypermultiplets  $n_H$  minus the number of vector multiplets  $n_V$  remains constant,  $\Delta(n_H - n_V) = 0$ .

5) Upon further toroidal compactification to four dimensions,  $D = 6$  tensor multiplets give rise to  $D = 4, N = 2$  vector multiplets whereas hypermultiplets and vector multiplets remain being so. In addition, extra vector multiplets (often named  $T, U$ ) associated to the torus and one containing the dilaton ( $S$ ) appear in the spectrum. Notice that in  $D = 4$  there is a Coulomb phase associated to vector multiplets, since the latter now contain scalars. This observation is relevant when studying the duality between Type-IIA compactified in a Calabi-Yau (CY) and the heterotic string compactified on  $K3 \times T^2$ .

Gauge and gravitational anomaly cancellation restricts very strongly the possible dynamics in  $D = 6, N = 1$  theories. Cancellation of the pure  $R^4$  gravitational anomalies requires [20, 21]:

$$n_H - n_V = 273 - 29n_T \quad (2.2)$$

where  $n_{H,V,T}$  are respectively the number of hyper, vector and tensor multiplets. Notice the fact that one tensor multiplet contributes as much as 29 hypermultiplets to the gravitational anomalies. This turns out to play an important role in non-perturbative transitions. If there is a simple gauge group  $G_a$ , cancellation of the pure  $F_a^4$  anomaly requires [22] :

$$T_a = \sum_i n_i t_a(R_i) \quad (2.3)$$

where  $n_i$  denotes the number of hypermultiplets transforming in the  $R_i$  representation, and the sum runs over the different representations.  $T_a, t_i(R_i)$  are defined by

$$\begin{aligned} \text{Tr } F_a^4 &= T_a \text{tr } F_a^4 + U_a (\text{tr } F_a^2)^2 \\ \text{tr } F_a^4 &= t_a(R_i) \text{tr } F_a^4 + u_a(R_i) (\text{tr } T_a^2)^2 \end{aligned} \quad (2.4)$$

where  $\text{Tr}$  ( $\text{tr}$ ) indicates trace in the adjoint (fundamental). Due to the absence of an independent fourth order Casimir, there are no quartic gauge anomalies for the exceptional groups and for  $SU(2)$  and  $SU(3)$ . For the classical groups  $SU(N)$ ,  $SO(N)$  and  $Sp(N)$ , one has [22]  $T_a = 2N$ ,  $(N-8)$  and  $(N+8)$  respectively <sup>1</sup>. Concerning  $t_a$ , one has  $t_a = 1$ ,  $(N-8)$  and  $\frac{1}{2}(N^2 - 17N + 54)$  for the fundamental, 2-index and 3-index antisymmetric representations for all classical groups <sup>2</sup>. For the spinorial representations of  $SO(2M)$  groups one has  $t_a = -2^{(M-5)}$ . Notice that cancellation of gauge and gravitational anomalies are consistent with the transitions:

$$\begin{aligned} 1 \text{ tensor} &\leftrightarrow \mathbf{28} + 1 \text{ hypermultiplet} \\ SO(8) + 1 \text{ tensor} &\leftrightarrow 1 \text{ hypermultiplet} \end{aligned} \quad (2.5)$$

where the **28** may be any anomaly free representation of dimension 28 like e.g. the 2-index antisymmetric representation of the  $N=8$  classical groups, or one half hypermultiplet in a **56** of  $E_7$ , etc. It turns out that these kind of transitions are physically realized by certain non-perturbative phenomena, as we will describe below. Once the pure quartic gravitational and gauge anomalies cancel, the anomaly polynomial  $A_8$

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<sup>1</sup>Notice that the group  $SO(8)$  is special: it is chiral in  $D=6$  but anomaly free in the absence of hypermultiplets.

<sup>2</sup>Notice that the 2-index antisymmetric representation is anomaly free for  $SU(8)$ ,  $SO(8)$  and  $Sp(8)$ .

which is left takes the form:

$$\begin{aligned}
A_8 &= \left(1 - \frac{n_T - 1}{8}\right)(\text{tr } R^2)^2 - \text{tr } R^2 \sum_a \tilde{C}_a \text{tr } F_a^2 + \\
&+ \sum_a \tilde{U}_a (\text{tr } F_a^2)^2 + \sum_{a < b} Y_{ab} \text{tr } F_a^2 \text{tr } F_b^2
\end{aligned} \tag{2.6}$$

where the values of the coefficients  $\tilde{C}_a$ ,  $\tilde{U}_a$  and  $Y_{ab}$  may be found in ref.[22]. For our present purposes it is only necessary to recall that for the case of a single tensor multiplet,  $n_T = 1$ , one can rewrite  $A_8$  in the factorized form:

$$A_8 = (\text{tr } R^2 - \sum_a V_a \text{tr } F_a^2)(\text{tr } R^2 - \sum_a \tilde{V}_a \text{tr } F_a^2) \tag{2.7}$$

where  $V_a$  is a gauge group factor which only depends on the group and is equal to 2, 1, 1/3, 1/3, 1/6 and 1/30 for  $SU(N)$ ,  $SO(N)$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$ , respectively. On the other hand,  $\tilde{V}_a$  depends in addition on the representations of the hypermultiplets. In particular, one finds for example [22] :

$$\begin{aligned}
\tilde{V}_{SU(N)} &= n_{a2} + (N - 4)n_{a3} + \frac{1}{2}(N - 4)(N - 5)n_{a4} - 2 \quad (for \ N \geq 4) \\
\tilde{V}_{SO(2N)} &= 2^{(N-6)}n_s - 2 \quad (for \ N \geq 3) \\
\tilde{V}_{E_6} &= \frac{1}{6}(n_{27} - 6) ; \quad \tilde{V}_{E_7} = \frac{1}{6}(n_{56} - 4) ; \quad \tilde{V}_{E_8} = \frac{-1}{5}
\end{aligned} \tag{2.8}$$

where  $n_{ai}$ ,  $i = 2, 3, 4$  is the number of hypermultiplets in the  $i$ -index antisymmetric representation and  $n_s$  is the number of spinorial representations. The  $A_8$  anomaly above has the appropriate form to be cancelled by the exchange of the unconstrained antisymmetric field  $B_{\mu\nu}$  present in the  $n_T = 1$  case. Notice that these values of the  $\tilde{V}_a$ 's correspond to the case with just one tensor multiplet. When several tensors are present there is a modified factorization of  $A_8$ , with  $\tilde{V}_a$  depending on  $n_T$  and an extra (perfect square) piece in  $A_8$  which involves only the gauge fields and is proportional to  $(n_T - 1)$ . Then, a generalized Green-Schwarz mechanism is at work [19], in which this extra piece is cancelled by the exchange of  $(n_T - 1)$  tensor multiplets. If a  $U(1)$  gauge theory is present, one has for the group theory coefficients in eq.(2.6)  $\tilde{C} = \frac{1}{6}\text{tr } Q^2$  and  $\tilde{U} = \frac{2}{3}\text{tr } Q^4$  [22] . Now, standard factorization as in eq.(2.7) is obtained only if  $\tilde{U} = \tilde{C} - 1$ . It turns out that in plenty of  $D = 6$ ,  $N = 1$  vacua this condition is not verified, even in perturbative models. In those cases what actually happens is that the

$U(1)$  is spontaneously broken and swallows one of the  $B$ -field modes with indices in the compact dimensions [20, 23, 24].

Due to supersymmetry, the anomaly coefficients  $V_a, \tilde{V}_a$  are related to the kinetic terms of vector multiplets. In particular, in the  $n_T = 1$  case one finds [19] :

$$L_{gauge}^{D=6} = -\frac{(2\pi)^3}{8\alpha'}\sqrt{G}\left(V_\alpha e^{-\phi/2} + \tilde{V}_\alpha e^{\phi/2}\right)\text{tr} F_\alpha^2 \quad (2.9)$$

where  $\phi$  is the real scalar in the unique tensor multiplet present, i.e., the dilaton. This will turn out to be relevant when we discuss heterotic/heterotic duality .

## 2.2 Perturbative Heterotic Vacua

### i) Toroidal $Z_N$ Orbifolds

We turn now to consider explicit  $D = 6, N = 1$  heterotic vacua. An interesting class of  $D = 6, N = 1$  heterotic vacua can be obtained from symmetric toroidal orbifold compactifications on  $T^4/Z_M$ . The construction of these models parallels that of  $T^6/Z_M$  orbifolds [25, 26] as considered in refs. [21, 22, 27]. Here we briefly review the notation and the salient points relevant to our discussion. Acting on the (complex) bosonic transverse coordinates, the  $Z_M$  twist  $\theta$  has eigenvalues  $e^{2\pi i v_a}$ , where  $v_a$  are the components of  $v = (0, 0, \frac{1}{M}, -\frac{1}{M})$ .  $M$  can take the values  $M = 2, 3, 4, 6$ . The embedding of  $\theta$  on the gauge degrees of freedom is usually realized by a shift  $V$  (not to be mistaken with the  $V_a$  coefficients that we introduced in the previous subsection!) such that  $MV$  belongs to the  $E_8 \times E_8$  lattice  $\Gamma_8 \times \Gamma_8$  or to the  $Spin(32)/Z_2$  lattice  $\Gamma_{16}$ . This shift is restricted by the modular invariance constraint

$$M(V^2 - v^2) = \text{even} \quad (2.10)$$

All possible embeddings for each  $M$  are easily found. In the  $E_8 \times E_8$  case, we find 2 inequivalent embeddings for  $Z_2$ , 5 for  $Z_3$ , 12 for  $Z_4$  and 59 for  $Z_6$ , leading to different patterns of  $E_8 \times E_8$  breaking to rank 16 subgroups. For  $Spin(32)/Z_2$  we find 3 inequivalent embeddings for  $Z_2$ , 5 for  $Z_3$ , 14 for  $Z_4$  and 50 for  $Z_6$ . Each of these models is only the starting point of a bigger class of vacua, generated by adding Wilson lines in the form of further shifts in the gauge lattice satisfying extra modular invariance constraints, by permutations of gauge factors, etc..

The spectrum for each model is subdivided in sectors. There are  $M$  sectors twisted by  $\theta^j$ ,  $j = 0, 1, \dots, M-1$ . Each particle state is created by a product of left and right vertex operators  $L \otimes R$ . At a generic point in the four-torus moduli space, the massless states follow from

$$m_R^2 = N_R + \frac{1}{2}(r + jv)^2 + E_n - \frac{1}{2} \quad ; \quad m_L^2 = N_L + \frac{1}{2}(P + jV)^2 + E_j - 1 \quad (2.11)$$

Here  $r$  is an  $SO(8)$  weight with  $\sum_{i=1}^4 r_i = \text{odd}$  and  $P$  a gauge lattice vector with  $\sum_{I=1}^{16} P^I = \text{even}$ .  $E_j$  is the twisted oscillator contribution to the zero point energy and it is given by  $E_j = j(M-j)/M^2$ . The multiplicity of states satisfying eq. (2.11) in a  $\theta^j$  sector is given by the appropriate generalized GSO projections [27]. In the untwisted sector there appear the gravity multiplet, a tensor multiplet, charged hypermultiplets and 2 neutral hypermultiplets (4 in the case of  $Z_2$ ). In the twisted sectors only charged hypermultiplets appear. The generalized GSO projections are particularly simple in the  $Z_2$  and  $Z_3$  case since essentially all massless states survive with the same multiplicity. The spectra for all  $Z_2$  and  $Z_3$  embeddings are shown in Tables 1 and 2 (from ref.[28]).

## ii) Smooth K3 Compactifications

It is instructive to compare these orbifold vacua with the  $D = 6$ ,  $N = 1$  models obtained upon generic heterotic compactifications on smooth  $K3$  surfaces in the presence of instanton backgrounds [29, 14, 4]. In the  $E_8 \times E_8$  case there are instanton numbers  $(k_1, k_2)$  satisfying  $k_1 + k_2 = 24$ , as required by anomaly cancellation. It is convenient to define  $k_1 = 12 + n$ ,  $k_2 = 12 - n$  and assume  $n \geq 0$  without loss of generality. For  $n \leq 8$ , an  $SU(2)$  background on each  $E_8$  leads to  $E_7 \times E_7$  unbroken gauge group with hypermultiplet content

$$\frac{1}{2}(8+n)(\mathbf{56}, \mathbf{1}) + \frac{1}{2}(8-n)(\mathbf{1}, \mathbf{56}) + 62(\mathbf{1}, \mathbf{1}) \quad (2.12)$$

Due to the pseudoreal character of the  $\mathbf{56}$  of  $E_7$ , odd values of  $n$  can also be considered. For the models in the range  $8 < n \leq 11$ , non-perturbative small instanton considerations are needed (see 2.3). There is a last model for  $n = 12$ , which is obtained by embedding all 24 instantons on one  $E_8$ . The reader may check how all the irreducible terms in the anomaly polynomial cancel and factorization of the residual anomalies occurs.



Shift $V$	Untwisted matter	Twisted matter	$(k_1, k_2)$
Group			
$\frac{1}{2}(1, 1, 0, \dots, 0) \times (0, \dots, 0)$	$(56, 2) + 4(1, 1)$	$8(56, 1) + 32(1, 2)^*$	$(24, 0)$
$E_7 \times SU(2) \times E_8$			
$\frac{1}{2}(1, 0, \dots, 0) \times (1, 1, 0 \dots, 0)$	$(1, 56, 2) + 4(1, 1, 1)$ $+ (128, 1, 1)$	$8(16, 1, 2)$	$(16, 8)$
$SO(16) \times E_7 \times SU(2)$			
$\frac{1}{3}(1, 1, 0, \dots, 0) \times (0, \dots, 0)$	$(56, 1) + 3(1, 1)$	$9(56, 1) + 18(1, 1)^*$ $+ 45(1, 1)^*$	$(24, 0)$
$E_7 \times U(1) \times E_8$			
$\frac{1}{3}(2, 0, \dots, 0) \times \frac{1}{3}(2, 0 \dots, 0)$	$(14, 1) + (64, 1) +$ $(1, 14) + (1, 64) + 2(1, 1)$	$9(14, 1) + 9(1, 14)$ $+ 18(1, 1)^*$	$(12, 12)$
$SO(14) \times SO(14) \times U(1)^2$			
$\frac{1}{3}(1, 1, 1, 1, 2, 0, 0, 0) \times (0, \dots, 0)$	$(84, 1) + 2(1, 1)$	$9(36, 1) + 18(9, 1)^*$	$(24, 0)$
$SU(9) \times E_8$			
$\frac{1}{3}(1, 1, 2, 0, \dots, 0) \times \frac{1}{3}(1, 1, 0 \dots, 0)$	$(27, 3, 1) + (1, 1, 56)$ $+ 3(1, 1, 1)$	$9(27, 1, 1) + 9(1, 3, 1)$ $+ 18(1, 3, 1)^*$	$(18, 6)$
$E_6 \times SU(3) \times E_7 \times U(1)$			
$\frac{1}{3}(1, 1, 1, 1, 2, 0 \dots, 0) \times \frac{1}{3}(1, 1, 2, 0, 0, 0)$	$(1, 27, 3) + (84, 1, 1)$ $+ 2(1, 1, 1)$	$9(9, 1, 3)$	$(15, 9)$
$SU(9) \times E_6 \times SU(3)$			

Table 1: Perturbative  $Z_2$  and  $Z_3$ ,  $E_8 \times E_8$ , orbifold models. The asterisk indicates twisted states involving left-handed oscillators. The last column shows which smooth  $K3$  compactification yields a similar massless spectrum *after Higgsing*.

Shift $V$	Untwisted matter	Twisted matter	$G_0$
Group			
$\frac{1}{2}(1, 1, 0, \dots, 0)$	$(28, 2, 2) + 4(1, 1, 1)$	$8(28, 1, 2) + 32(1, 2, 1)^*$	$SO(8)$
$SO(28) \times SU(2) \times SU(2)$			
$\frac{1}{2}(1, 1, 1, 1, 1, 1, 0, \dots, 0)$	$(12, 20) + 4(1, 1)$	$8(32, 1)$	$SO(8)$
$SO(12) \times SO(20)$			
$\frac{1}{4}(1, \dots, 1, -3)$	$(120) + (\overline{120})$ $+ 4(1)$	$8(16) + 8(\overline{16})$	1
$SU(16) \times U(1)$			
$\frac{1}{3}(1, 1, 0, \dots, 0)$	$(28, 2) + 3(1, 1)$	$9(28, 2) + 18(1, 1)^*$ $+ 45(1, 1)^*$	$SO(8)$
$SO(28) \times SU(2) \times U(1)$			
$\frac{1}{3}(1, 1, 1, 1, 1, 2, 0, \dots, 0)$	$(22, 5) + (1, 10)$ $+ 2(1, 1)$	$9(22, 1) + 9(1, 10)$ $+ 18(1, 5)^*$	$SO(8)$
$SO(22) \times SU(5) \times U(1)$			
$\frac{1}{3}(1, 1, 1, 1, 1, 1, 1, 1, 0, \dots, 0)$	$(16, 8) + (1, 28)$ $+ 2(1, 1)$	$9(1, 28) + 18(1, 1)^*$	$SO(8)$
$SO(16) \times SU(8) \times U(1)$			
$\frac{1}{3}(1, \dots, 1, 2, 0, 0, 0, 0)$	$(10, 11) + (1, 55)$ $+ 2(1, 1)$	$9(1, 11) + 9(16, 1)$	1
$SO(10) \times SU(11) \times U(1)$			
$\frac{1}{3}(1, \dots, 1, 0, 0)$	$(14, 2, 2) + (91, 1, 1)$ $+ 2(1, 1, 1)$	$9(1, 1, 1) + 9(14, 2, 1)$ $+ 18(1, 1, 2)^*$	1
$SU(14) \times U(1) \times SU(2) \times SU(2)$			

Table 2: Perturbative  $Z_2$  and  $Z_3$ ,  $Spin(32)/Z_2$ , orbifold models . The asterisk indicates twisted states involving left-handed oscillators. The last column shows the generic terminal gauge group  $G_0$  after Higgsing.

Models with diverse groups can be obtained from these spectra by symmetry breaking. The group from the second  $E_8$  does not possess, in general, enough charged matter to be completely broken. Higgsing stops at some terminal group, depending on the value of  $n$ , with minimal or no charged matter [29, 27]. For instance  $E_8$ ,  $E_7$ ,  $E_6$ ,  $SO(8)$ ,  $SU(3)$  terminal groups are obtained for  $n = 12, 8, 6, 4, 3$  while complete breaking proceeds for  $n = 2, 1, 0$ . On the other hand, the first  $E_7$  can be completely Higgsed away. For generic gauge group  $G = G_1 \times G_2$  with  $G_1$  and  $G_2$  subgroups of the first and second  $E_8$  obtained from backgrounds with instanton numbers  $(12 + n, 12 - n)$ , the following identity is satisfied

$$\frac{\tilde{V}_1}{V_1} = \frac{n}{2} \quad ; \quad \frac{\tilde{V}_2}{V_2} = -\frac{n}{2} \quad (2.13)$$

These relations remain valid at each step of possible Higgsing. From the anomaly polynomial it follows that the gauge kinetic terms are proportional to [19]

$$-V_1(e^{-\phi} + \frac{n}{2}e^{\phi})\text{tr } F_1^2 - V_2(e^{-\phi} - \frac{n}{2}e^{\phi})\text{tr } F_2^2 \quad (2.14)$$

where  $F_i$  is the field strength of the unbroken group  $G_i$  and  $\phi$  is the scalar dilaton living in a  $D = 6$  tensor multiplet. The coefficient of the gauge kinetic term for the second  $E_8$  is such that the gauge coupling diverges at [14, 4]

$$e^{-2\phi} = \frac{n}{2} \quad (2.15)$$

This is a sign of a phase transition in which there appear tensionless strings [3, 4, 5], as we will describe in section 3.5.

In the last column of Table 1 we show the instanton numbers  $(k_1, k_2)$  of compactifications yielding, upon Higgsing, a massless spectrum similar to the corresponding orbifold. We thus see that the five  $Z_3$  orbifolds of  $E_8 \times E_8$  are in the same moduli space as generic  $K3$  compactifications with  $n = 12, 0, 12, 6, 3$  respectively. The two  $Z_2$  orbifolds correspond to  $n = 12, 4$  respectively. This connection between modular invariant orbifold models and instanton backgrounds is explained in more detail in ref.[28] .

In the  $Spin(32)/Z_2$  case, embedding a total instanton number  $k = 24$  is required to cancel gravitational anomalies. An  $SU(2)$  background breaks the symmetry down to  $SO(28) \times SU(2)$  with hypermultiplets in  $10(\mathbf{28}, \mathbf{2}) + 65(\mathbf{1}, \mathbf{1})$ . Hence, upon Higgsing,

the generic group is  $SO(8)$ . This class of models is known to be [8, 23] in the same moduli space as  $(k_1, k_2) = (16, 8)$  compactifications of  $E_8 \times E_8$ . As shown in Table 2, the first three  $Z_3$  orbifolds of  $Spin(32)/Z_2$  do have  $SO(8)$  as generic group but the last two models have trivial gauge group after full Higgsing. In fact, it was already noticed in [24] that the fourth  $Spin(32)/Z_2$   $Z_3$  model could lead to complete Higgsing. Also, in ref. [23] the authors construct a heterotic  $Z_2$  orbifold, ‘without vector structure’, in which the resulting  $U(16)$  group can be completely broken. In our language this  $Z_2$  orbifold has embedding  $V = \frac{1}{4}(1, \dots, 1, -3)$  (third example in Table 2). In general, orbifold embeddings with vector structure have shifts  $V$  such that  $MV = (n_1, \dots, n_{16})$ , whereas embeddings without vector structure have  $MV = (n_1 + \frac{1}{2}, \dots, n_{16} + \frac{1}{2})$ . Since  $MV \in \Gamma_{16}$ ,  $\sum_I n_I = \text{even}$  in both cases.

We have seen that the  $E_8 \times E_8$  compactifications can be labeled by a pair of instanton numbers  $(k_1, k_2)$  with  $k_1 = 12 + n$ ,  $k_2 = 12 - n$  and  $n = 0, \dots, 12$ . Recently it has become clear that there are in fact different types of  $Spin(32)/Z_2$  instantons which are classified by the generalized second Stieffel-Whitney class [23]. An analysis in terms of F-theory [30] has shown that in a general  $Spin(32)/Z_2$  heterotic compactification, instantons with and without vector structure are present, their contribution to the total instanton number being respectively  $8 + 4n$  and  $16 - 4n$ , with the integer  $n$  satisfying  $-2 \leq n \leq 4$ . A simple heterotic realization of this idea can be obtained by embedding a  $U(1) \times SU(2)$  background in  $SO(32) \supset SU(16) \times U(1) \supset SU(14) \times U(1)' \times U(1) \times SU(2)$ . Then the  $Spin(32)/Z_2$  vacua can be labeled by giving the pair of instanton numbers  $(k_{NA}, k_A)$  with  $k_{NA} = 8 + 4n$  and  $k_A = 16 - 4n$ . The adjoint decomposition is

$$\begin{aligned}
\mathbf{496} = & (\mathbf{1}, 0, 0, \mathbf{3}) + (\overline{\mathbf{14}}, \frac{1}{2}, 0, \mathbf{2}) + (\mathbf{14}, -\frac{1}{2}, 0, \mathbf{2}) + (\mathbf{195}, 0, 0, \mathbf{1}) + 2(\mathbf{1}, 0, 0, \mathbf{1}) + \\
& (\mathbf{1}, 1, \frac{1}{2\sqrt{2}}, \mathbf{1}) + (\mathbf{14}, \frac{1}{2}, \frac{1}{2\sqrt{2}}, \mathbf{2}) + (\mathbf{91}, 0, \frac{1}{2\sqrt{2}}, \mathbf{1}) + \\
& (\mathbf{1}, -1, -\frac{1}{2\sqrt{2}}, \mathbf{1}) + (\overline{\mathbf{14}}, -\frac{1}{2}, -\frac{1}{2\sqrt{2}}, \mathbf{2}) + (\overline{\mathbf{91}}, 0, -\frac{1}{2\sqrt{2}}, \mathbf{1})
\end{aligned} \tag{2.16}$$

where the two middle entries denote the  $U(1)' \times U(1)$  charges. The massless spectrum that arises upon embedding  $k_A = (16 - 4n)$  instantons in  $U(1)$  and  $k_{NA} = (8 + 4n)$  in  $SU(2)$  is found using the index theorem formulae [20, 24]. For  $-1 \leq n \leq 2$  we find the

following  $SU(14) \times U(1)' \times U(1)$  hypermultiplets

$$\begin{aligned}
& (1 - \frac{n}{2})(\mathbf{1}, 1, \frac{1}{2\sqrt{2}}) + (1 - \frac{n}{2})(\mathbf{1}, -1, -\frac{1}{2\sqrt{2}}) + (1 - \frac{n}{2})(\mathbf{91}, 0, \frac{1}{2\sqrt{2}}) + \\
& (1 - \frac{n}{2})(\overline{\mathbf{91}}, 0, -\frac{1}{2\sqrt{2}}) + (6 + n)(\overline{\mathbf{14}}, -\frac{1}{2}, -\frac{1}{2\sqrt{2}}) + (6 + n)(\mathbf{14}, \frac{1}{2}, \frac{1}{2\sqrt{2}}) + \\
& (2 + 2n)(\mathbf{14}, -\frac{1}{2}, 0) + (2 + 2n)(\overline{\mathbf{14}}, \frac{1}{2}, 0) + (33 + 8n)(\mathbf{1}, 0, 0)
\end{aligned} \tag{2.17}$$

For  $n = 3$  there are not enough instantons to support the  $U(1)$  bundle. The corresponding instantons become small and give the spectrum of a pointlike instanton without vector structure [30] (see section 2.3). The resulting model has a gauge group  $SO(28) \times SU(2) \times Sp(4)$ , a hypermultiplet content

$$8(\mathbf{28}, \mathbf{2}, \mathbf{1}) + 56(\mathbf{1}, \mathbf{1}, \mathbf{1}) + \frac{1}{2}(\mathbf{28}, \mathbf{1}, \mathbf{8}) + (\mathbf{1}, \mathbf{2}, \mathbf{8}) \tag{2.18}$$

and one additional tensor multiplet. For  $n = 4$ , instantons without vector structure disappear and one just has the  $SU(2)$  bundle with 24 instantons mentioned above. For  $n = -2$ , the situation is reversed, since there only remains a  $U(1)$  bundle with 24 instantons. The resulting gauge group is  $U(16)$  with hypermultiplets

$$2(\mathbf{120}, \frac{1}{2\sqrt{2}}) + 2(\overline{\mathbf{120}}, -\frac{1}{2\sqrt{2}}) + 20(\mathbf{1}, 0) \tag{2.19}$$

For each value of  $n$ , appropriate sequential Higgsing produces chains of models that match similar  $E_8 \times E_8$  heterotic chains [27], for the same value of  $n$ , thus providing several identifications between compactifications of both heterotic strings. This equivalence is evident in the F-theory framework (see chapter 4), since the Calabi-Yau spaces obtained upon Higgsing (taking generic polynomials in the fibration over  $\mathbb{F}_n$ ) are identical in both types of chains. By computation of  $\tilde{V}/V$  it can be shown that the  $Z_3$  models listed in Table 2 correspond to  $n = 4, 4, 4, 1, 1$ , respectively. The three  $Z_2$  models correspond to  $n = 4, 4, 0$ .

### iii) T-Dualities Between $E_8 \times E_8$ and $Spin(32)/Z_2$ , $D = 6$ , $N = 1$ Vacua

As discussed above,  $E_8 \times E_8$  and  $Spin(32)/Z_2$  compactifications corresponding to the same values of  $|n|$  are in the same moduli space. This means that they must be in some way  $T$ -dual. This  $T$ -duality may be explicitly shown in some cases in terms of orbifold compactifications as we now discuss.  $T$ -duality in toroidal compactifications is

already present in  $D = 9$  [31]. More concretely, one can show that the  $E_8 \times E_8$  heterotic compactified in a circle of radius  $R$  and in the presence of a Wilson line of the form  $a = \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0)(1, 1, 1, 1, 0, 0, 0, 0)$ , is equivalent to the  $Spin(32)/Z_2$  compactified in a circle with radius  $1/R$  and Wilson line  $a = \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0)$ . In both cases the gauge group is  $SO(16) \times SO(16)$  and the spectrum and interactions are identical.

Things are not so immediate in  $D = 6$ ,  $N = 1$  theories since we have now a smaller number of supersymmetries and the gauge backgrounds (e.g., Wilson lines) are not arbitrary. However, some of the equivalences can still be easily proven. The equivalences for the cases  $n = 4$  and  $n = 0$  were shown in terms of  $Z_2$  orbifolds in ref.[23]. We rephrase their discussion in the language of the bosonic formulation of the heterotic string.

#### *T-duality of $n = 4$ vacua*

Consider a  $E_8 \times E_8$   $Z_2$  orbifold with shift  $V = \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0)(1, 1, 0, 0, 0, 0, 0, 0)$ . This breaks the symmetry down to  $SO(16) \times E_7 \times SU(2)$ . As shown in Table 1, this is an orbifold version of a  $(k_1, k_2) = (16, 8)$   $E_8 \times E_8$  compactification (hence  $n = 4$ ). Add now a quantized Wilson line  $a = \frac{1}{2}(0, 0, 1, 1, 1, 1, 0, 0)(1, 1, 1, 1, 0, 0, 0, 0)$  which is of the same type discussed above for the  $D = 9$  case. It is easy to check that these verify the modular invariance constraints. The unbroken gauge group is  $SO(8) \times SO(8) \times SO(12) \times SO(4)$ . Consider now the  $Z_2$ ,  $Spin(32)/Z_2$  orbifold with shift  $V = \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0)$ . This breaks the symmetry down to  $SO(12) \times SO(20)$ . As shown in Table 1 upon full Higgsing this model (like the  $E_8 \times E_8$  one) leads to a generic  $SO(8)$  group (hence  $n = 4$ ). Add now the Wilson line  $a = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$  which is again of the same type as above. Again, this is a modular invariant model with the same gauge group. It is easy to check that the massless and massive spectra of these two models are exactly the same. Thus the  $n = 4$   $E_8 \times E_8$  compactification and the  $Spin(32)/Z_2$  compactifications with vector structure are in fact  $T$ -dual.

#### *T-duality of $n = 0$ vacua*

A  $n = 0$  vacuum in  $E_8 \times E_8$  is symmetric in both groups, so we have to construct

now an orbifold model with this symmetry. Consider first a  $E_8 \times E_8$ ,  $Z_2$  orbifold with shift  $V = \frac{1}{4}(-3, 1, 1, 1, 1, 1, 1, 1)(1, 1, 1, 1, 1, 1, 1, 1)$ . It is easy to see that this shift is equivalent in the  $E_8 \times E_8$  lattice to the second one in Table 1, leading to an unbroken  $SO(16) \times E_7 \times SU(2)$  gauge group. This is not symmetric in both  $E_8$ 's but the model may be symmetrized [15] by adding a discrete Wilson line of the same form as in the previous example,  $a = \frac{1}{2}(1, -1, -1, -1, 0, 0, 0, 0)(-1, -1, -1, 1, 0, 0, 0, 0)$ . Now the gauge group is  $U(8) \times U(8)$ . There are hypermultiplets in the untwisted sector transforming like  $(\mathbf{28} + \overline{\mathbf{28}}, \mathbf{1}) + (\mathbf{1}, \mathbf{28} + \overline{\mathbf{28}}) + 4(\mathbf{1}, \mathbf{1})$  and in the twisted sectors like  $8(\mathbf{8} + \overline{\mathbf{8}}, \mathbf{1}) + 8(\mathbf{1}, \mathbf{8} + \overline{\mathbf{8}})$ . The reader may check that this is indeed a  $n = 0$  model by recalling that  $n = 2\tilde{V}/V$  and noting that  $\tilde{V} = 2 - 2 = 0$  from eqs.(2.8). Now, one can show that the same model may be constructed starting from the  $Spin(32)/Z_2$  orbifold without vector structure obtained from the  $Z_2$  shift  $V = \frac{1}{4}(-3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ . If we add again the Wilson line  $a = \frac{1}{2}(1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0)$ , the unbroken group is again  $U(8) \times U(8)$  and the hypermultiplet content is identical to the previous  $E_8 \times E_8$  case. This is not surprising since both models are subject to the same Wilson line and the gauge group before the  $Z_2$  twist is identical in both cases ( $SO(16)^2$ ). In addition, the shift  $V$ , when restricted to this subgroup in both cases, is also identical. So this shows that the  $n = 0$   $E_8 \times E_8$  compactifications are T-dual to  $Spin(32)/Z_2$  compactifications without vector structure.

These  $n = 0$  vacua are relevant for heterotic/heterotic duality in  $D = 6$ , as we will discuss later on. In fact,  $n = 2$  vacua present also heterotic/heterotic duality. It has been shown by using F-theory that both type or vacua are equivalent. This would suggest that they are in some sense  $T$ -dual. We will discuss this point further in chapter 4.

## 2.3 Non-Perturbative Heterotic Vacua and Small Instantons

### i) Small Instantons in $Spin(32)/Z_2$ Heterotic

Consider first a standard  $Spin(32)/Z_2$  heterotic compactification on a *smooth*  $K3$ . As indicated above, a consistent perturbative background requires the presence of a

total of 24 instantons. However, when the size of one of these instantons becomes small something interesting happens [32]. The perturbative compactification with  $k = 23$  is anomalous. However, the heterotic dilaton diverges exactly at the location of the small instanton, no matter how small its asymptotic value is [33], so one has to deal with strongly coupled dynamics. In this case, these dynamics yield a new non-perturbative gauge symmetry  $Sp(1) \simeq SU(2)$ , along with hypermultiplets transforming as  $\frac{1}{2}(\mathbf{32}, \mathbf{2}) + (\mathbf{1}, \mathbf{1})$  with respect to the group  $SO(32) \times Sp(1)$ . What happens is that at this point a Type-I Dirichlet five-brane appears with precisely those world-volume fields<sup>3</sup>. The world-volume fills the six uncompactified dimensions, so these fields appear in spacetime. If we decompose the hypermultiplets with respect to the unbroken subgroup of  $SO(32)$ , one can check that gauge and gravitational anomalies cancel. Thus beyond perturbation theory the condition  $k = 24$  is replaced by

$$k + n_B = 24 \tag{2.20}$$

where  $n_B$  is now the number of dynamical  $SO(32)$  five-branes, which can be understood as small instantons [32]. One such brane carries, as we said, an  $Sp(1)$  vector multiplet, but when  $r$  of them coincide at a point on the smooth  $K3$ , the group is enhanced to  $Sp(r)$ . In general, the non-perturbative group is  $\prod Sp(r_i)$  with  $\sum r_i = n_B$ . The five-branes also carry non-perturbative hypermultiplets. In particular, for each  $Sp(r)$  there appear 32 half hypermultiplets in the fundamental representation, together with one hypermultiplet in the antisymmetric two-index representation (decomposable as a singlet plus the rest). Cancellation of gauge anomalies requires that the hypermultiplets in the fundamental representation to be also charged under the perturbative gauge group that arises when  $SO(32)$  is broken by the background with instanton number  $k = 24 - n_B$ .

This is just a particular example of a more general phenomenon. Further possibilities appear if the small instanton sits not at a smooth  $K3$  point but at an A-D-E orbifold singularity. In the case of the  $SO(32)$  heterotic string, their dynamics corresponds to that of Type-I D-branes at singular points, which have been recently studied

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<sup>3</sup>Actually the dynamical object is composed of *two* Type-I D-branes, with  $SU(2)$  Chan-Paton factors.



in [34, 35, 36], based on previous results from ref. [37]. Five-branes with a variety of world-volume field contents appear in this case. Some examples [28] corresponding to  $A_m$ ,  $m = 1, \dots, 5$  singularities are displayed in Table 3. Notice that in these cases, except for the case of  $Z_2$  singularities without vector structure, the world-volume theories contain tensor multiplets, and not only vector multiplets as in the smooth K3 case. Thus small  $Spin(32)/Z_2$  instantons on singularities give transitions to Coulomb phases parametrized by the real scalars in tensor multiplets.

Recently non-perturbative  $D = 6$  heterotic orbifold models have also been constructed which correspond to the presence of small instantons either moving on the bulk or stuck at the orbifold fixed points [28]. These are in some sense toroidal orbifold versions of the K3 vacua with  $k < 24$  considered above. We present an example here which contains five-branes stuck at  $Z_2$  singularities. It is a  $Z_2$  orbifold of heterotic  $SO(32)$  which yields the same spectrum as the  $Z_2$  orientifold constructed by Dabholkar and Park [38] and model C of Gopakumar and Mukhi [39]. This is a  $D = 6$ ,  $N = 1$  model with gauge group  $SO(8)^8$ , seventeen tensors and four hypermultiplets. It can be obtained in terms of F-theory compactified on the standard  $Z_2 \times Z_2$  orbifold, as a compactification of M-theory on  $T^5/Z_2 \times Z_2$  and as a Type-IIB orientifold. Here we will obtain it as a heterotic  $SO(32)$   $Z_2$  orbifold (with a non-modular invariant gauge shift [28]). We will embed the  $Z_2$  twist in terms of a shift  $V$  in the  $\Gamma_{16}$  lattice supplemented with two discrete Wilson lines  $a_1$  and  $a_2$  as follows

$$\begin{aligned} V &= a_1 = \frac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0) \\ a_2 &= \frac{1}{2}(0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) \end{aligned} \quad (2.21)$$

The two Wilson lines break the symmetry down to  $SO(8)^4$  whereas the  $V$  shift projects out all charged multiplets from the untwisted sector. Only the four untwisted singlet hypermultiplets remain in that sector. The sixteen twisted sectors split into four sets of four fixed points each, which are subject to shifts  $V$ ,  $V + a_2$ ,  $V + a_1 + a_2$  and  $V + a_1$  respectively. The first three sets of four fixed points are all similar, the corresponding shift has eight  $\frac{1}{2}$  entries. Thus  $V^2 = 2$  and there are no massless hypermultiplets at any of those twelve fixed points. Looking at table 3, we see that for  $Z_2$  embeddings with vector structure one tensor and a gauge group  $Sp(\ell) \times Sp(\ell + \frac{\ell-1}{2} - 4)$  appear [34].

$Z_M$	Gauge Group	Hypermultiplets	$n_T$
Embeddings with vector structure			
–	$Sp(\ell)$	$\frac{32}{2}(2\ell) + \ell(2\ell - 1)$	0
$Z_2$	$Sp(\ell) \times Sp(\ell + \frac{w_1}{2} - 4)$	$w_0(2\ell, 1) + w_1(1, 2\ell + w_1 - 8) + (2\ell, 2\ell + w_1 - 8)$	1
$Z_2$	$Sp(\ell) \times SO(2\ell + 8) \quad [w_1 = 0]$	$(2\ell, 2\ell + 8)$	1
$Z_3$	$Sp(\ell) \times U(2\ell + w_1 - 8)$	$w_0(2\ell, 1) + w_1(1, 2\ell + w_1 - 8)$ $+(2\ell, 2\ell + w_1 - 8) + (1, (\ell + \frac{w_1}{2} - 4)(2\ell + w_1 - 9))$	1
$Z_4$	$Sp(\ell) \times U(2\ell + w_1 + w_2 - 8)$ $\times Sp(\ell + \frac{w_1}{2} + w_2 - 8)$	$w_0(2\ell, 1, 1) + w_1(1, 2\ell + w_1 + w_2 - 8, 1)$ $+w_2(1, 1, 2\ell + w_1 + 2w_2 - 16) + (2\ell, 2\ell + w_1 + w_2 - 8, 1)$ $+(1, 2\ell + w_1 + w_2 - 8, 2\ell + w_1 + 2w_2 - 16)$	2
Embeddings without vector structure			
$Z_2$	$U(2\ell)$	$\frac{32}{2}(2\ell) + 2(\ell(2\ell - 1))$	0
$Z_4$	$U(2\ell) \times U(2\ell + u_2 - 8)$	$u_1(2\ell, 1) + u_2(1, 2\ell + u_2 - 8) + (2\ell, 2\ell + u_2 - 8)$ $+(1, (\ell + \frac{u_2}{2} - 4)(2\ell + u_2 - 9)) + (\ell(2\ell - 1), 1)$	1
$Z_6$	$U(2\ell) \times U(2\ell + u_2 + u_3 - 8)$ $\times U(2\ell + u_2 + 2u_3 - 16)$	$u_1(2\ell, 1, 1) + u_2(1, 2\ell + u_2 + u_3 - 8, 1) + (\ell(2\ell - 1), 1, 1)$ $+u_3(1, 1, 2\ell + u_2 + 2u_3 - 16) + (2\ell, 2\ell + u_2 + u_3 - 8, 1)$ $+(1, 2\ell + u_2 + u_3 - 8, 2\ell + u_2 + 2u_3 - 16)$ $+(1, 1, (\ell + \frac{u_2}{2} + u_3 - 8)(2\ell + u_2 + 2u_3 - 17))$	2

Table 3: Some world-volume theories of  $SO(32)$  five-branes at  $Z_M$  singularities . Here  $w_\mu$  is the number of entries equal to  $\frac{\mu}{M}$  in  $V$  with vector structure. Similarly,  $u_\mu$  is the number of entries equal to  $\frac{2\mu-1}{2M}$  in  $V$  without vector structure.

Since in our case  $w_1 = 8$ , we get one tensor for each of the twelve fixed points and no enhanced gauge group for  $\ell = 0$ .

The other four fixed points with shift  $V + a_1$  have a different behaviour. Indeed, this shift is trivial and hence we have  $w_1 = 0$  for those fixed points. As remarked in ref. [36], five-branes at a  $Z_2$  singularity with  $w_1 = 0$  give transitions to a Coulomb phase with one tensor multiplet and a gauge group  $Sp(\ell) \times SO(2\ell + 8)$  (see Table 3). Thus, in our case, with  $\ell = 0$  at each of the fixed points we have altogether a non-perturbative group  $SO(8)^4$  and four tensor multiplets. Putting all the contributions together we get the total content  $SO(8)^8$ , seventeen tensor multiplets and four singlet hypermultiplets. Notice how the 16 twisted sectors are in a Coulomb phase, twelve of them with  $w_1 = 8$  yielding only tensors and the other four have  $w_1 = 0$  yielding in addition the required non-perturbative  $SO(8)^4$ .

## ii) Small Instantons in $E_8 \times E_8$

The physics of small instantons in  $E_8 \times E_8$  heterotic is somewhat different since the strongly coupled limit of this theory is given by M-theory compactified on the segment  $S^1/Z_2$ . Now the M-theory five-branes play the role played by type I Dirichlet five-branes in the  $SO(32)$  case. When compactifying the  $E_8 \times E_8$  heterotic on a smooth K3 with  $n_B$  pointlike instantons,  $n_B$  five-branes appear, with their world-volume spanning the six uncompactified dimensions. The world-volume theory includes one  $D = 6$  tensor multiplet and one singlet hypermultiplet. Altogether, each one contains 5 real scalars which parametrize the position of the five-brane on  $K3 \times S^1/Z_2$ . The five-branes are a source of torsion so that in a case with  $k_1$  instantons in the first  $E_8$ ,  $k_2$  in the second and  $n_B$  five-branes at points in  $K3 \times S^1/Z_2$ , the condition  $k_1 + k_2 = 24$  is replaced by

$$k_1 + k_2 + n_B = 24 \tag{2.22}$$

For *smooth* K3 compactifications, the physics when the instanton becomes small is very different in  $E_8 \times E_8$  compared to the  $SO(32)$  case. In the latter case the transition may be considered as a standard Higgs phase in the sense that the process in which the instanton recovers a finite size may be described as a Higgs effect in which the non-perturbative gauge symmetry  $Sp(n_B)$  is Higgsed away. Also, the  $D = 6$  theory living on the five-brane world-volume is infrared free. In the  $E_8 \times E_8$  case the small instanton

theory is at a Coulomb phase parametrized by the real scalar in the tensor multiplet [3, 4]. At the transition point, unlike the  $SO(32)$  case, there is a non-trivial scale invariant interacting  $D = 6$  field theory in which tensionless strings appear. Away from the transition point there is a massless tensor multiplet plus a hypermultiplet which, as we said, parametrizes the position of the five-brane.

The above remarks seem to indicate that there are no enhanced non-perturbative gauge groups in  $E_8 \times E_8$  compactifications. However, this statement has to be qualified in various ways. First, we know that  $E_8 \times E_8$  compactifications with  $0 \leq n \leq 4$  are T-dual to  $Spin(32)/Z_2$  compactifications with the same  $n$ , and we know that in such  $Spin(32)/Z_2$  compactifications *there are* non-perturbative enhanced gauge groups. One thus concludes that  $E_8 \times E_8$  K3 smooth compactifications with  $0 \leq n \leq 4$  may have indeed non-perturbative enhanced gauge symmetries in points of their moduli space corresponding to small instantons in the T-dual  $Spin(32)/Z_2$  model [23]. The second qualification concerns the effect of K3 singularities. We already remarked how in the  $Spin(32)$  case five-branes sitting at singularities give rise not only to additional non-perturbative gauge groups but also to tensors. The opposite can be said in  $E_8 \times E_8$ : when five-branes sit at A-D-E singularities not only tensors appear but also new non-perturbative gauge groups. This has been recently analyzed in [40].

### 3 Type-IIA $D = 6, 4$ Vacua and Type-II/Heterotic Duality

#### 3.1 $D = 4, N = 2$ Type-II/Heterotic vacua

In this and the following subsections we will make a detour from six dimensions to study  $D = 4, N = 2$  heterotic and Type-IIA vacua, and the relations among them.

The supermultiplet structure of  $D = 4, N = 2$  theories is

$$\begin{aligned}
Gravity &\rightarrow R(4) = \{g_{\mu\nu}, \psi_\mu^{(+)}, \psi_\mu^{(-)}, B_\mu\} & ; & \quad \mu, \nu = 0, \dots, 3 \\
Vector &\rightarrow V(4) = \{A_\mu, \lambda^{(+)}, \lambda^{(-)}, 2a\} \\
Hypermultiplet &\rightarrow H(4) = \{\chi^{(-)}, \chi^{(+)}, 4\phi\} .
\end{aligned} \tag{3.1}$$

The presence of two kinds of scalars, those in vector multiplets and those in hypermultiplets, implies the existence of two branches in the moduli space. The Higgs branch is parametrized by the scalars in hypermultiplets, and is very similar to the six dimensional Higgs branch. Vevs for charged hypermultiplets trigger gauge symmetry breaking with reduction (in general) of the rank. At a generic point in this phase, the gauge group is broken to a terminal one, which usually does not contain any charged matter. The Coulomb phase, on the other hand, is associated to the scalars in vector multiplets. Along this branch, gauge symmetry breaking of the non-abelian groups occurs, due to vevs associated to the scalars in the adjoint. Consequently, there is no rank reduction, and at a generic point in this moduli space the symmetry group reduces to the Cartan subalgebra. Also, mass terms are generated for the charged hypermultiplets, due to their coupling to the scalars in vector multiplets, and they become heavy.

Notice that  $D = 6$ ,  $N = 1$  theories reduce to  $D = 4$ ,  $N = 2$  ones upon compactification on a  $T^2$ . In this reduction, the multiplets (2.1) decompose as

$$\begin{aligned}
\text{Gravity} \quad R(6) &\longrightarrow R(4) + 2V(4) \\
\text{Tensor} \quad T(6) &\longrightarrow V(4) \\
\text{Vector} \quad V(6) &\longrightarrow V(4) \\
\text{Hyperm.} \quad H(6) &\longrightarrow H(4) .
\end{aligned} \tag{3.2}$$

Notice that the  $D = 6$  tensor multiplet goes over to a  $D = 4$  vector multiplet, so that the  $D = 6$  Coulomb branch we introduced in previous sections is naturally contained in the (larger)  $D = 4$  Coulomb branch.

These  $D = 4$ ,  $N = 2$  theories are non-chiral, and thus not subject to anomaly cancellation constraints. However, strong statements can still be made concerning their non-perturbative behaviour due to some non-renormalization theorems, analogous to the six dimensional ones, in the sense that the scalars in vector- and hypermultiplets are decoupled.

Heterotic  $D = 4$ ,  $N = 2$  vacua are obtained through compactification on  $K3 \times T^2$ , along with the choice of a non-trivial gauge bundle over the internal space. In some cases this construction amounts simply to a compactification to six dimensions on  $K3$

followed by a reduction on  $T^2$  to  $D = 4$ . The spectrum in this case is easily found by a decomposition of the  $D = 6$  spectrum (obtained as in section 2.2) following (3.3). It is important to notice that in the  $T^2$  compactification an additional  $U(1)^4$  factor appears, from the graviphoton in the gravity multiplet, the tensor multiplet containing the dilaton, and the two vector multiplets associated to the torus moduli. Also, one gets an additional  $U(1)$  for each extra tensor multiplet in the six dimensional theory. Upon decompactification of the torus, one recovers the initial  $D = 6$  model.

There are other models which cannot be understood this way, since their construction involves choosing a  $T^2$  with its Kähler class frozen at a specific value [29]. The spectrum can be found through index theorems as well. Note that in this case, the decompactification of the torus is not possible, and the models are not related to  $D = 6$ ,  $N = 1$  constructions. In this sense, they are intrinsically four dimensional.

In both cases, the generic spectrum of the  $D = 4$ ,  $N = 2$  heterotic theory on the Coulomb branch contains an abelian gauge symmetry  $U(1)^{n_v+1}$  and  $n_H$  neutral hypermultiplets.

Type-IIA  $D = 4$ ,  $N = 2$  vacua can be obtained by compactification on a Calabi-Yau threefold. The spectrum is easily found by Kaluza-Klein reduction of the  $D = 10$  Type-IIA supergravity. If  $(h_{11}, h_{12})$  denote the Hodge numbers of the internal space, the gauge group is  $U(1)^{h_{11}+1}$  (where the  $h_{11}$  vector bosons are obtained by integration of the ten dimensional 3-form on the  $h_{11}$  non-trivial 2-cycles, and the remaining vector is the graviphoton), and there are  $h_{12} + 1$  hypermultiplets (where  $h_{12}$  come from integrating the 3-form over  $h_{12}$  3-cycles, and the remaining one contains the Type-IIA dilaton). These hypermultiplets are neutral with respect to the symmetry group. Note that this construction seems to be purely four dimensional, so in principle one would not expect  $D = 6$ ,  $N = 1$  relatives of these models. We will see this is not the case if the CY space is elliptically fibered.

### 3.2 $D = 4$ , $N = 2$ Type-II/Heterotic duality

At a generic point in the Coulomb branch, the spectrum of a heterotic  $D = 4$ ,  $N = 2$  compactification is very analogous to the kind of spectra found for Type-IIA com-

pactifications on Calabi-Yau spaces. Actually a non-perturbative duality relation has been conjectured [41, 29] to hold between both kinds of compactifications. A necessary condition for two models to be dual is the matching of their spectra [29]:

$$\begin{aligned} n_V &= h_{11} \\ n_H &= h_{12} + 1 \end{aligned} \tag{3.3}$$

It has also been determined [42, 43] that the Calabi-Yau should be a  $K3$  fibration, so that the base  $\mathbb{P}_1$  is a preferred  $(1,1)$ -cycle, which gives the mode dual to the heterotic dilaton. This fact allows for a very intuitive picture of the duality *via* the adiabatic argument outlined in [44]. Understanding the heterotic  $K3 \times T^2$  as a fibration of  $T^4$  over  $\mathbb{P}_1$ , and the Type-IIA Calabi-Yau as a fibration of  $K3$  over  $\mathbb{P}_1$ , the  $D = 4$  duality is obtained by fiberwise application of the six dimensional duality between the heterotic string on  $T^4$  and the Type-IIA on  $K3$ . This  $D = 6$ ,  $N = 2$  equivalence is firmly established and better understood than the proposed  $D = 4$ ,  $N = 2$  ones, so one can hope that some features of the former persist in the latter, helping in understanding their richer dynamics.

For example, heterotic compactifications lead to enhanced non-abelian gauge symmetries whenever vevs for some scalars are set to zero. The Type-IIA mechanism required for generating such symmetries is inherited from an analogous phenomenon in the  $D = 6$ ,  $N = 2$  theory: a singularity develops on the internal manifold, so that Type-IIA 2-branes wrapping the zero size 2-cycles give additional vector multiplets enhancing a product of  $U(1)$  factors to a full non-abelian symmetry group.

A strong check of the duality between the heterotic and Type-IIA theories is the construction of dual pairs verifying (3.3) [29, 41, 44, 45]. In some cases, the heterotic sides of several dual pairs appear to be connected by perturbative processes (typically the Higgs mechanism). Duality then requires that non-perturbative transitions must exist among the Type-IIA realizations of these models. Our aim in this chapter is to accumulate evidence in favour of this matching between the web of heterotic and Type-IIA vacua, and to learn about the non-perturbative dynamics it reveals.

An interesting subset of the moduli space can be explored by the construction of chains of models as in [27]. The idea can be easily carried out in terms of the bundle

construction of section 2.2. Consider the  $D = 4$ ,  $N = 2$  reduction of the  $D = 6$ ,  $N = 1$  spectrum (2.12). As already mentioned, we can Higgs the second  $E_7$  down to a terminal group. We can then Higgs the first  $E_7$  sequentially, lowering the rank of the gauge symmetry in units, and going to the Coulomb branch at each stage. This generates a chain of models whose spectra can be compared, *via* (3.3), with the Hodge numbers of candidate Calabi-Yau duals.

Let us consider for example the  $n = 4$  case, for which the spectrum (2.12), once in  $D = 4$ ,  $N = 2$ , is

$$E_7 \times E_7 \times U(1)^4$$

$$6(\mathbf{56}, \mathbf{1}) + 2(\mathbf{1}, \mathbf{56}) + 62(\mathbf{1}, \mathbf{1}) \quad (3.4)$$

The second  $E_7$  can be Higgsed down to  $SO(8)$  with no matter. By going to the Coulomb phase, we get a model with a gauge symmetry  $U(1)^{15}$  and 69 neutral hypermultiplets. The CY space reproducing this spectrum must have, from the matching conditions,  $(h_{11}, h_{12}) = (14, 68)$ . Instead, we can go further along the Higgs branch, by the sequential breaking of the remaining  $E_7$  to  $E_6$ ,  $SO(10)$ ,  $SU(5)$ ,  $SU(4)$ ,  $SU(3)$ ,  $SU(2)$  and to nothing. The Hodge numbers of candidate dual CY's are found to be  $(13, 79)$ ,  $(12, 88)$ ,  $(11, 95)$ ,  $(10, 122)$ ,  $(9, 153)$ ,  $(8, 194)$  and  $(7, 271)$ , respectively. The easiest way of identifying such spaces is to look for these Hodge numbers in the lists of CY three-folds which are  $K3$  fibrations. Actually, in the tables of [42] one finds candidate CY's for the last four elements in the chain, realized as the varieties  $\mathbb{P}_5^{(1,1,4,8,10,12)}$ [20, 16],  $\mathbb{P}_4^{(1,1,4,8,10)}$ [24],  $\mathbb{P}_4^{(1,1,4,8,14)}$ [28] and  $\mathbb{P}_4^{(1,1,4,12,18)}$ [36].

This exercise can be carried out for other values of  $n$ <sup>4</sup>. The Calabi-Yau varieties associated to the last steps of Higgsing for  $n=2,4,6,8,12$  can be found in the tables of ref. [42]. Remarkably, a pattern in the weights defining these spaces is observed. The heterotic cascade breaking sequence

$$\dots \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow \emptyset \quad (3.5)$$

---

<sup>4</sup>A subtlety arises for  $n = 9, 10, 11$ , in which case there are not enough instantons in the second  $E_8$  to support an  $SU(2)$  bundle, so they are forced to be point-like.



maps into the following sequence in the Type-II side

$$\begin{aligned} \mathbb{P}_5^{(1,1,w_1,w_2,w_3,w_4)} &\rightarrow \mathbb{P}_4^{(1,1,w_1,w_2,w_3)} \rightarrow \\ \mathbb{P}_4^{(1,1,w_1,w_2,w_3+w_1)} &\rightarrow \mathbb{P}_4^{(1,1,w_1,w_2+w_1,w_3+2w_1)} \end{aligned} \quad (3.6)$$

Moreover, these transitions can be recast in terms of  $n$ , as follows

$$\begin{aligned} \mathbb{P}_5^{(1,1,n,n+4,n+6,n+8)}[2n+12, 2n+8] &\rightarrow \mathbb{P}_4^{(1,1,n,n+4,n+6)}[3n+12] \rightarrow \\ \mathbb{P}_4^{(1,1,n,n+4,2n+6)}[4n+12] &\rightarrow \mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12] \end{aligned} \quad (3.7)$$

In the following section the construction of these CY's will be considered in a more appropriate framework, which allows to find CY spaces for any value of  $n$  in the range  $0 \leq n \leq 12$ , and moreover shows that this pattern remains valid for all  $n$ .

These regularities point towards the existence of transitions among the compactifications on these CY's. [27, 24]. They are generalizations of the conifold transition in [46, 47]. One starts from a CY corresponding to a model in the Coulomb branch. By setting scalars in vector multiplets to zero, some 2-cycles on the CY collapse, leading to a singularity that produces a non-abelian enhancement of the symmetry. The singularity can then be smoothed by a deformation of the complex structure, i.e. by turning on vevs for scalars in hypermultiplets. This process is the dual picture of the sequential Higgsing we have introduced from the heterotic viewpoint.

There also exist processes connecting chains for different values of  $n$  [4]. They consist, in the heterotic picture, on shrinking an  $E_8$  instanton to zero size, and transforming it into a M-theory five-brane. Changing the vev of the scalar in its tensor multiplet the five-brane can be made to travel along the Coulomb branch until it is reabsorbed as an instanton on the other  $E_8$ . This process is purely six dimensional and moreover, cannot be described within the framework of field theory, since it is mediated by tensionless strings [3]. Note that the transformation of five-branes into finite size instantons realizes the first transition proposed in (2.5). The tensor present in the Coulomb phase turns into hypermultiplets (in adequate representations under the gauge group) in the Higgs phase.

The construction of chains can also be carried out from the heterotic  $SO(32)$  bundle construction introduced in section 2.2. This time  $n$  defines how the instanton number

24 splits between instantons with and without vector structure and has the range  $-2 \leq n \leq 4$ . Starting from the unbroken group  $SU(14) \times U(1)' \times U(1)$  it is possible to Higgs the abelian factors, and start a sequential breaking of  $SU(14)$ . Using the spectra determined in section 2.2, it can be checked that for  $-2 \leq n \leq 2$  complete Higgsing is possible, while for  $n = 4$  one ends up with a  $SO(8)$  without matter. These coincide with the groups found in the  $E_8 \times E_8$  case for the same  $n$ . Actually, this coincidence is also obtained for the previous elements in the Higgsing chains, and can be understood as a consequence of the T-dualities between both heterotics, as advanced in section 2.2. For  $n = 3$ , on the other hand, the initial spectrum contains an additional tensor, associated to a small instanton. However, it can be argued from F-theory that coincidence with the  $E_8 \times E_8$  spectrum is found once the tensor disappears non-perturbatively.

The mechanism we have described for changing the value of  $n$  is also valid in the  $SO(32)$  case. The essential point is the appearance of a tensor degree of freedom at some locus in the hypermultiplet moduli space, as has been mentioned in section 2.3. It has been checked from F-theory that one can travel along this Coulomb branch, and land on a different Higgs branch, where the tensor disappears and there has been a change of one unit in  $n$  [30]. The correct interpretation of this phenomenon in the  $SO(32)$  heterotic is challenging, since there is no direct relation to M-theory. It certainly point towards a more unified picture of both heterotics.

### 3.3 F-Theory-Heterotic Duality

In the discussion of the previous subsection the heterotic models we employed where essentially six dimensional, the dynamics did not depend on the  $T^2$ , while the Type-IIA construction is defined directly in four dimensions. The existence of a well defined decompactification limit imposes some condition on the geometry of the CY spaces, namely they must be elliptic fibrations. This has been determined from the Type-IIA viewpoint [48], but we will motivate it from a different approach, F-theory [7, 8, 9].

### 3.3.1 Introduction to F-theory

A new insight into several string dualities has been provided by F-theory [7], a construction that can be understood as a Type-IIB compactification on a variety  $B$  in the presence of Dirichlet seven-branes. The complex ‘coupling constant’  $\tau = a + ie^{-\varphi/2}$ , where  $a$  is the RR scalar and  $\varphi$  is the dilaton field, depends on space-time and is furthermore allowed to undergo  $SL(2, \mathbb{Z})$  monodromies around the seven-branes. This  $\tau$  can be thought to describe the complex structure parameter of a torus (of frozen Kähler class, since Type-IIB theory has no fields to account for it) varying over the compactifying space  $B$ , and degenerating at the eight-dimensional submanifolds defined by the world-volumes of the seven-branes. The constraint of having vanishing first Chern class (the contribution of the seven-branes cancelling that of the manifold  $B$ ) forces the fibration of  $T^2$  over  $B$  thus constructed to be an elliptic CY manifold  $X$ . Thus, F-theory compactifications are defined only on elliptically fibered manifolds.

The compactification of F-theory on the product of such an elliptically fibered manifold  $X$  and a circle  $S^1$ , lies on the same moduli space as M-theory compactified on  $X$  [7]. The result follows from the fact that M-theory on  $T^2$  is equivalent to the Type-IIB theory on  $S^1$ , the IIB coupling constant  $\tau$  being equal to the modular parameter of the M-theory torus. By adiabatic fibering of this duality over the base  $B$  of the manifold  $X$  of interest, the desired result is obtained. This equivalence has proved fruitful in encoding string dualities in lower dimensions, and, especially, in clarifying several phenomena in heterotic string compactifications.

### 3.3.2 F-Theory/Heterotic duality

After compactification on an elliptic  $K3$ , F-theory gives a  $D = 8$  theory conjectured to be dual to the heterotic string compactified on  $T^2$  [7, 49, 50]. The mapping of the moduli between both constructions is as follows. The size of the base  $\mathbb{P}_1$  is related to the heterotic dilaton whereas the 18 polynomial deformation complex parameters of the fibration match the heterotic toroidal Kähler and complex structure moduli together with Wilson line backgrounds. The Kähler class of the elliptic fiber on  $K3$  has no physical meaning in F-theory, and thus, no heterotic counterpart.

Fibering this model over another  $\mathbb{P}_1$  gives a family of F-theory compactifications on CY three-folds which are  $K3$  fibrations, with the  $K3$  fibers admitting an elliptic fibration structure. The resulting base spaces are the Hirzebruch surfaces  $\mathbb{F}_n$ , which are fibrations of  $\mathbb{P}_1$  over  $\mathbb{P}_1$ , characterized by an integer  $n$ . These models are naturally conjectured to be dual to heterotic string compactifications on  $K3$  ( $T^2$  fibered over  $\mathbb{P}_1$ ) with gauge bundles embedded on  $E_8 \times E_8$ . For some values of  $n$ , it can also be related to  $SO(32)$  heterotic string compactifications [23, 48], as we will mention at the end of the section. Upon toroidal compactification to  $D = 4$ ,  $N = 2$ , heterotic/Type-IIA duality is recovered so that F-theory provides an  $N = 1$ ,  $D = 6$  version of this duality. This implies that the CY spaces dual to essentially six dimensional heterotic models should be elliptically fibered, so that they can be used for F-theory compactification.

The  $D = 6$ ,  $N = 1$  spectrum obtained from compactifying F-theory on a threefold can be partially determined using this relationship with Type-IIA compactifications. If we denote by  $h_{11}(B)$  the number of  $(1,1)$ -forms of the base  $B$ , and  $(h_{11}(X), h_{12}(X))$  the Hodge numbers of  $X$ , one can show [7] that the number of tensor multiplets is  $h_{11}(B) - 1$ , the rank of the gauge group is  $h_{11}(X) - h_{11}(B) - 1$  and the number of neutral hypermultiplets is  $h_{12} + 1$ .

Notice that in  $D = 6$  there is no vector Coulomb branch, so the non-abelian gauge groups do not break to their Cartan subalgebra. Further information about the gauge group is encoded in the curves of singularities of  $X$  [8, 9], i.e. in the overlapping D7-branes, in the IIB language. Charged hypermultiplets also remain in the massless spectrum, and they are usually associated to the intersection of the curves of singularities, i.e. correspond to open-strings stretching between different D7-branes.

The base space we are interested in,  $\mathbb{F}_n$ , being a  $\mathbb{P}_1$  fibration over another  $\mathbb{P}_1$ , has two Kähler forms, and thus the massless spectrum contains only one tensor multiplet (associated to the heterotic dilaton [8]). Consequently, except when the singularities in the variety require a blow-up of the base for their resolution, we will have  $n_T = 1$ .

Our purpose is to find the F-theory duals of the previously discussed heterotic models [9], by explicit construction of the fibrations as hypersurfaces in projective varieties.

#### **i) F-theory and heterotic $E_8 \times E_8$**

The elliptic fiber can be realized as  $\mathbb{P}_2^{(1,2,3)}$  [6]. Introducing coordinates  $z_1, w_1$  and  $z_2, w_2$  for the two  $\mathbb{P}_1$ 's,  $x, y$  for the torus, and two  $C^*$  quotients  $\lambda, \mu$  to projectivize the affine spaces, we obtain the following ambient space

$$\begin{array}{cccccc} & z_1 & w_1 & z_2 & w_2 & x & y \\ \lambda : & 1 & 1 & 0 & 0 & 4 & 6 \\ \mu : & n & 0 & 1 & 1 & 2n+4 & 3n+6 \end{array} \quad (3.8)$$

The hypersurface in this space is given by the fibration equation

$$y^2 = x^3 + f(z_1, w_1; z_2, w_2)x + g(z_1, w_1; z_2, w_2) \quad (3.9)$$

where  $f$  and  $g$  are polynomials such that the equation is invariant under the  $C^*$  actions, and the variety is well defined on the projective space. Notice that for given  $\omega_i, z_i$  this equation describes a torus and the complete equation is thus an elliptic fibration. It can be shown that for  $n > 12$  the variety described by (3.8) and (3.9) does not fulfill the CY condition (in particular, the associated Newton polyhedron ceases to be reflexive), so that there are 13 possible spaces.

For a fixed value of  $n$  the moduli space of the fibrations (3.9), parametrized by the coefficients of the polynomials  $f, g$ , reproduces the Higgs branch of the heterotic  $E_8 \times E_8$  model corresponding to embedding  $(12+n, 12-n)$  instantons in  $E_8 \times E_8$ . Coulomb branches associated to tensors are obtained by blowing up the base  $\mathbb{F}_n$ .

The  $E_8 \times E_8$  structure is manifest for a particular choice of polynomials [9], leading to

$$\begin{aligned} y^2 = & x^3 + f_8(z_2, w_2)z_1^4w_1^4x \\ & + g_{12-n}(z_2, w_2)z_1^7w_1^5 + g_{12}(z_2, w_2)z_1^6w_1^6 + g_{12+n}(z_2, w_2)z_1^5w_1^7 \end{aligned} \quad (3.10)$$

This model has gauge group  $E_8 \times E_8$ , since  $z_1 = 0$  and  $z_1 = \infty$  ( $w_1 = 0$ ) are two curves of  $E_8$  singularities. The number of independent parameters can be computed to be 44. Also 24 blow-ups on the base space are required, so there are 24 tensor degrees of freedom. This model corresponds to the heterotic  $E_8 \times E_8$  with 24 pointlike instantons.

From this very special point in moduli space, one can reach more generic ones by allowing for more generic polynomials  $f, g$ . This gives finite size to the small instantons. Even though the intermediate steps are also interesting e.g. to understand how charged matter is encoded in the equations [51, 52] we will mainly center on the models obtained upon maximal Higgsing, i.e. on the most generic polynomials.

To this end we expand the polynomials  $f, g$  in powers of  $z_1, w_1$

$$\begin{aligned} f(z_1, w_1; z_2, w_2) &= \sum_{k=-4}^4 z_1^{4+k} w_1^{4-k} f_{8-nk}(z_2, w_2) \\ g(z_1, w_1; z_2, w_2) &= \sum_{l=-6}^6 z_1^{6+l} w_1^{6-l} g_{12-nl}(z_2, w_2) \end{aligned} \quad (3.11)$$

where subscripts denote the degree of the polynomial in  $z_2, w_2$  (only non-negative degrees are admitted).

For  $n \neq 0, 1$ , we can dehomogenize with respect to  $w_1$  using one of the  $C^*$  quotients, and the variety can be represented by the hypersurface  $\mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12]$ . These coincide with the last elements of the chains of section 3.2, showing they are elliptically fibered, as required. Furthermore, for *all* values of  $n$ , the Hodge numbers of the fibration do match the matter spectrum of heterotic models on  $K3 \times T^2$  with  $SU(2)$  bundles of  $(12+n, 12-n)$  instanton number embedded in  $E_8 \times E_8$ , upon maximal Higgsing and moving to the Coulomb phase [27, 51]. Thus, one identifies Type-IIA compactifications on these spaces as duals of the heterotic constructions in  $D = 4$ , or equivalently the F-theory compactifications as duals of the heterotic models in  $D = 6$  (decompactifying the  $T^2$ ).

Let us illustrate with an example which further checks can be performed to confirm that this construction provides the required CY's. Take the  $n = 4$  case in eqs. (3.11). The gauge group comes from the generic singularity type along  $w_1 = 0$ . Locally (i.e. to lowest order in  $w_1$ ) the fibration can be written

$$y^2 = x^3 + w_1^2 f_0 x + w_1^3 g_0 \quad (3.12)$$

which is a  $D_4$  singularity (the last term is an irrelevant deformation), giving rise to a  $SO(8)$  gauge symmetry. Moreover, there are no additional D7-branes intersecting the curve  $w_1 = 0$  ( $f_0$  is a constant) and consequently no charged matter. Also, the total

number of neutral hypermultiplets is obtained from  $h_{12}$ , and matches the heterotic result. A stronger check consists on counting independently the moduli associated to each initial  $E_8$ , and the  $K3$  moduli [52]. This computation yields a number in agreement with the index theorem for instantons embedded in  $E_8$  used in the heterotic construction. In particular, let us compute the number of independent monomial deformations for  $k, l < 0$  in (3.11): there are 76 monomials coming from  $f_{12}$ ,  $f_{16}$ ,  $f_{20}$ , and  $f_{24}$ , and 162 from  $g_{16}$ ,  $g_{20}$ ,  $g_{24}$ ,  $g_{28}$ ,  $g_{32}$ , and  $g_{36}$ . One must, however, subtract 5 which can be eliminated by redefinitions  $z_1 \rightarrow z_1 + P_4(z_2, w_2)$ , and a last one from a global scaling of the polynomial equation. The result, 232, is equal to  $30(12 + 4) - 248$ , the number of moduli for 16  $E_8$  instantons. Similar exercises show the mentioned agreement in all cases. Thus, every detail in the  $D = 6$  spectrum can be understood in the F-theory framework. It has also been shown that by restriction of the polynomial coefficients in (3.11) one can reproduce the CY spaces of the remaining elements in the chains, with results in complete agreement with heterotic expectations [51, 52].

The F-theory description is also fruitful in providing a detailed description of the  $D = 6$  transitions mediated by tensionless strings (see section 3.5). The simplest example which we have already encountered is the transition changing the value of  $n$ . In the F-theory context this amounts to a transition  $\mathbb{F}_n \rightarrow \mathbb{F}_{n\pm 1}$ , which is realized by blowing up the base at a point and then blowing down a curve of self-intersection  $(-1)$  [4, 9]. In the intermediate step a new Kähler class is introduced on the base, associated to the additional tensor from the M-theory five-brane. Also, at the boundaries of the Coulomb branch, a Type-IIB 3-brane wraps around the collapsed (1,1) cycles generating the tensionless string in  $D = 6$ .

### **F-theory and heterotic $Spin(32)/Z_2$**

The heterotic  $Spin(32)/Z_2$  admits a F-theory description as well, by means of elliptic fibrations with two sections, in order to have  $Spin(32)/Z_2$  (instead of  $SO(32)$ ) as the gauge group [48, 30]. These can be obtained by particularizing the polynomials  $f, g$  in (3.9) so that the fibration takes the factorized form

$$y^2 = (x - p(z_1, w_1; z_2, w_2))(x^2 + p(z_1, w_1; z_2, w_2)x + q(z_1, w_1; z_2, w_2)) \quad (3.13)$$

The heterotic model with unbroken  $SO(32)$  is reproduced for the particular choice of

polynomials [48]

$$\begin{aligned}
p(z_1, w_1; z_2, w_2) &= B_{4+2n}(z_2, w_2)w_1^4 \\
q(z_1, w_1; z_2, w_2) &= A_{8+4n}(z_2, w_2)w_1^8 - 4B_{4+2n}(z_2, w_2)C_{4-n}(z_2, w_2)w_1^5z_1^3 \\
&\quad - 2C_{4-n}(z_2, w_2)^2w_1^2z_1^6
\end{aligned} \tag{3.14}$$

It can be checked that  $w_1 = 0$  is a curve of  $D_{16}$  singularities, but making this evident requires a non-trivial change of variables. There also appear several  $Sp(k_i)$  factors, with  $\sum k_i = 8 + 4n$ , with matter content coinciding with that of  $SO(32)$  small instantons at smooth points (see table 3). Also  $4 - n$  blow-ups of the base are required at  $w_1 = 0$  and the zeroes of  $C_{4-n}(z_2, w_2)$ . This leads to  $4 - n$  tensors associated to the Kähler classes of the new curves, and a copy of  $Sp(4)$  (with some matter content) each, since the fibers over the exceptional divisors are singular. This contribution is generated from  $4 - n$  groups of four small instantons each, at  $Z_2$  singular points in the heterotic  $K3$  (see table 3). The number of neutral hypermultiplets comes up to be consistent with gravitational anomaly cancellation. This family of models illustrates how the different heterotic  $Spin(32)/Z_2$  compactifications, with splitting of instantons defined by  $n$ , arise in F-theory.

As in the  $E_8 \times E_8$  case, one can give the instantons a finite size by allowing for more generic polynomials in the fibration. These deformation fall in two classes, those which respect the factorized form (3.13) and those which do not. If one is to remain within theories with a  $SO(32)$  heterotic interpretation, one must be careful in turning on the latter, as we will show below. Let us for the time being expand the most generic polynomials of the first kind

$$\begin{aligned}
p(z_1, w_1; z_2, w_2) &= \sum_{k=-2}^2 p_{4-kn}(z_2, w_2)z_1^{2+k}w_1^{2-k} \\
q(z_1, w_1; z_2, w_2) &= \sum_{k=-4}^4 q_{8-kn}(z_2, w_2)z_1^{4+k}w_1^{4-k}
\end{aligned} \tag{3.15}$$

$$\tag{3.16}$$

stressing again they will *not* allow us to explore the whole of the F-theory moduli space. The models obtained for the most generic polynomials are closely similar to



those in the  $E_8 \times E_8$  case (actually, when one allows for non-factorizable deformations the spaces are identical). This fact was already noticed in the heterotic construction of section 2.2, and is behind the many T-dualities between compactifications of both heterotics on  $K3$ . For  $-2 \leq n \leq 2$  Higgsing can proceed completely, while for  $n = 4$  a terminal  $SO(8)$  without matter is found. The  $n = 3$  case is interesting, its local description near  $w_1 = 0$  being

$$y^2 = x^3 + (q_2(z_2, w_2) - p_1(z_2, w_2)^2)w_1^2 x - p_1(z_2, w_2)q_2(z_2, w_2)w_1^3 \quad (3.17)$$

which corresponds to a semi-split  $D_4$  singularity [52], leading to  $SO(7)$  with two spinorials. This does not coincide directly with the  $SU(3)$  without matter found in the  $E_8 \times E_8$  case, but matches it upon Higgsing.

As in the  $E_8 \times E_8$  case, all kind of transitions within a given chain, or between chains which differ in the value of  $n$  are possible, by application of the same geometrical operations in the corresponding CY's. As remarked at the end of section 4.2, this points towards the idea that both heterotic construction lead to identical models, and only differ in the interpretation of the moduli.

### 3.4 The A,B,C,D chains

The construction of the  $SU(2)$  bundle backgrounds in  $E_8 \times E_8$  and  $SO(32)$  heterotics has been a useful tool to understand a large class of vacua in the moduli space of  $D = 6, 4$  string vacua. We would like to stress the importance of the initial observation of the regularities in the weights of the CY's dual to some of these models.

This is enough motivation to search for similar patterns in the tables of [42]. These were already noticed in [27] and further analyzed both from the F-theory and heterotic points of view in [24]. The regularities found are shown in table 4, already recast in terms of an integer  $n$ . The family A corresponds to the chains studied in the previous section. Our purpose in the following is to understand the structure of these CY spaces, and to find their heterotic duals.

We first note that these CY's are  $K3$  fibrations, with the  $K3$ 's being elliptic fibrations. The  $T^2$  fibers are realized as  $\mathbb{P}_3^{(1,2,3)}$ [6],  $\mathbb{P}_3^{(1,1,2)}$ [4],  $\mathbb{P}_3^{(1,1,1)}$ [3] and  $\mathbb{P}_4^{(1,1,1,1)}$ [2, 2]

$r$	A	B
4	$\mathbb{P}_5^{(1,1,n,n+4,n+6,n+8)}$	
3	$\mathbb{P}_4^{(1,1,n,n+4,n+6)}$	$\mathbb{P}_5^{(1,1,n,n+2,n+4,n+6)}$
2	$\mathbb{P}_4^{(1,1,n,n+4,2n+6)}$	$\mathbb{P}_4^{(1,1,n,n+2,n+4)}$
1	$\mathbb{P}_4^{(1,1,n,2n+4,3n+6)}$	$\mathbb{P}_4^{(1,1,n,n+2,2n+4)}$
$r$	C	D
2	$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+4)}$	
1	$\mathbb{P}_4^{(1,1,n,n+2,n+2)}$	$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+2)}$

Table 4: Structure of the A,B,C and D chains.

for the A, B, C and D families, respectively. This means that the models can be made sense of in  $D = 6$  through compactification of F-theory on them.

The elliptic fibration structure can be analyzed following the steps studied for the A models. For example, the CY's of the B family are defined in the ambient space given by

$$\begin{array}{cccccc}
& z_1 & w_1 & z_2 & w_2 & x & y \\
\lambda & 1 & 1 & 0 & 0 & 2 & 4 \\
\mu & n & 0 & 1 & 1 & n+2 & 2n+4
\end{array} \tag{3.18}$$

and the hypersurface is provided by the equation

$$y^2 = x^4 + f(z_1, w_1; z_2, w_2) x^2 + g(z_1, w_1; z_2, w_2) x + h(z_1, w_1; z_2, w_2) \tag{3.19}$$

In each case  $n$  is restricted by the condition that the set of weights lead to a well defined CY space. For type A,  $n \leq 12$  in agreement with the heterotic construction. For types B and C, the weights correspond to reflexive polyhedra only for  $n \leq 8$  and  $n \leq 6$  respectively. For models D,  $n \leq 4$  is expected.

Let us consider some general features in their spectra. In table 5 (from ref.[24]) we show the Hodge numbers of the last elements in the chains. The bases of the

fibrations are in most cases Hirzebruch surfaces  $\mathbb{F}_n$ , but in the cases signalled with an asterisk some blow-ups of the base are required. So in general we get just one tensor multiplet. This fact and the existence of the discrete parameter  $n$  suggest there may exist a heterotic construction in terms of bundles embedded in  $E_8 \times E_8$ , with  $n$  defining how the instanton number splits.

$n$	A	B	C	D
0	(243,3)	(148,4)	(101,5)	(70,6)
1	(243,3)	(148,4)	(101,5)	(70,6)
2	(243,3)	(148,4)	(101,5)	(70,6)
3	(251,5)	(152,6)	(103,7)	(70,10)*
4	(271,7)	(164,8)	(111,9)	(76,10)
5	(295,7)	(178,10)	(120,12)*	
6	(321,9)	(194,10)	(131,11)	
7	(348,10)	(210,12)*		
8	(376,10)	(227,11)		
9	(404,14)*			
10	(433,13)*			
11	(462,12)*			
12	(491,11)			

Table 5: Hodge numbers  $(b_{21}^1, b_{11}^1)$  for the terminal spaces.

We also see there is a general feature in the Hodge numbers in table 5. For a given  $n$ , as one follows the sequence  $A \rightarrow B \rightarrow C \rightarrow D$  (when possible)  $h_{11}$  increases typically in one unit, while  $h_{12}$  decreases in diverse amounts (which also follow certain numerological patterns, which we omit for the sake of brevity). The natural explanation is that the new families have an enhanced gauge symmetry: this increases the rank of the gauge group (related to  $h_{11}$ ) and lowers the number of neutral hypermultiplets

(related to  $h_{12}$ ), since some previously neutral hypermultiplets can become charged with respect to the new symmetry.

The last piece of information comes from a detailed analysis of the fibration equations. Following the analysis performed for the family A of models, one can obtain the terminal gauge groups upon maximal Higgsing. The gauge group singularities exist only for  $n \geq 2$  and are located at the curve  $w_1 = 0$ , just like in the case of the family A. The actual groups usually coincide with the terminal groups for chains A for the same  $n$ . One can also count the number of moduli coming from each (conjectured) bundle in  $E_8$ . This computation reveals the nature of the underlying groups before breaking through instantons takes place. For example, the chains B yield  $[18(9+n)-133]$  moduli for the gauge factor which is completely broken, pointing out the existence of an initial  $E_7$  gauge symmetry. Indeed, the dimension and Coxeter number of  $E_7$  are 133 and 18 respectively. Similarly,  $E_6$  and  $SO(10)$  structures underlay C and D models, respectively. Observe that there seems to be some missing instanton number, also reflected in the smaller range of possible  $n$ 's, suggesting it must somehow be associated to the generation of the enhanced symmetry. Just to finish, concerning the nature of this extra groups, we are forced to accept they are not non-abelian, otherwise would have been detected as curves of singularities on the CY spaces. Abelian factors, however, appear in a much more elusive way, related to the rank of the Mordell-Weyl group of the elliptic fibration [9].

This information serves as a guide for the construction of the heterotic duals. We will show how the desired new  $D = 6$  heterotic models can be most readily obtained by considering generic  $H \times U(1)^{8-d}$  backgrounds in each  $E_8$ , with  $H$  some non-Abelian factor [24].

For example, embedding  $SU(2) \times U(1)$  backgrounds with instanton numbers  $(k_1, m_1; k_2, m_2)$  in both  $E_8$ 's gives the following  $E_8 \times U(1) \times E_8 \times U(1)$  spectrum

$$\begin{aligned} & \left\{ \frac{1}{6}(3k_1 + m_1 - 12)(\mathbf{27}, \frac{1}{2\sqrt{3}}; \mathbf{1}, 0) + \right. \\ & \frac{1}{6}(3k_2 + m_2 - 12)(\mathbf{1}, 0; \mathbf{27}, \frac{1}{2\sqrt{3}}) + \\ & \left. \frac{1}{3}(m_1 - 3)(\mathbf{27}, -\frac{1}{\sqrt{3}}; \mathbf{1}, 0) + \right. \end{aligned}$$

$$\begin{aligned}
& \frac{1}{3}(m_2 - 3)(\mathbf{1}, 0; \mathbf{27}, -\frac{1}{\sqrt{3}}) + \\
& \frac{1}{2}(k_1 + 3m_1 - 4)(\mathbf{1}, \frac{\sqrt{3}}{2}; \mathbf{1}, 0) + \\
& \frac{1}{2}(k_2 + 3m_2 - 4)(\mathbf{1}, 0; \mathbf{1}, \frac{\sqrt{3}}{2}) + \text{c.c.} \} + \\
& ((2k_1 - 3) + (2k_2 - 3) + 20)(\mathbf{1}, 0; \mathbf{1}, 0)
\end{aligned} \tag{3.20}$$

In this case gravitational anomalies cancel as long as  $k_1 + m_1 + k_2 + m_2 = 24$ .

At this point a brief comment concerning  $U(1)$  anomalies is in order. It is easy to check that  $U(1)$ 's in this class of theories are in general anomalous. More precisely, one finds that the anomaly 8-form  $I_8$  does not generically factorize into a product of two 4-forms, so that the Green-Schwarz mechanism cannot cancel the residual anomaly. Instead one finds that the linear combination of  $U(1)$  charges

$$Q_f = \cos \theta Q_1 + \sin \theta Q_2 \tag{3.21}$$

leads to a factorized  $I_8$  as long as

$$\sin^2 \theta = \frac{m_2}{m_1 + m_2} \quad ; \quad \cos^2 \theta = \frac{m_1}{m_1 + m_2} \tag{3.22}$$

independently of the values of  $k_{1,2}$ . Thus, for given  $m_{1,2}$ , there is a linear combination of both  $U(1)$ 's which is anomaly-free but the orthogonal combination is not. Thus, somehow, the latter combination must be spontaneously broken. Indeed, a mechanism by which this can take place was suggested in refs. [53, 20] for analogous compactifications. The idea is that in  $D = 10$  the kinetic term of the  $B_{MN}$  field contains a piece

$$H^2 \simeq (\partial_\mu B_{ij} + A_\mu^1 \langle F_{ij}^1 \rangle + A_\mu^2 \langle F_{ij}^2 \rangle)^2 \tag{3.23}$$

where the indices  $i, j$  live in the four compact dimensions. Notice that one linear combination of  $A_\mu^1$  and  $A_\mu^2$  will become massive by swallowing a  $B_{ij}$  zero mode.

From eq.(3.20) one notes that in the presence of the  $SU(2)$  bundles the values of  $m_{1,2}$  are forced to be multiples of 3 in order to have half-integer numbers of  $(\mathbf{27} + \overline{\mathbf{27}})$  and also  $m_{1,2} \geq 3$ . Thus the simplest class of models of this type will have instanton numbers  $(k_1, 3; k_2, 3)$  and the unbroken  $U(1)_f$  is in this case the diagonal combination

$U(1)_D$ . The fact that  $k_1 + k_2 = 18$ , instead of  $k_1 + k_2 = 24$  (as in the case without  $U(1)$  backgrounds), hints at the required heterotic duals of models of type B. Indeed, in these models, the range for the values of  $n$  is smaller ( $n \leq 8$ ) and this is probably the case here since the range for  $k_{1,2}$  is also smaller. Moreover, models B have a number of vector multiplets one unit higher compared to the corresponding chain A elements. This is precisely the case here, due to the presence of the extra  $U(1)_D$ . These arguments are compelling enough to consider this sort of heterotic constructions in more detail. Let us sketch how upon sequential Higgsing of the non-Abelian symmetries the spectrum in (3.20) one does in fact reproduce chains of type B. In analogy with the usual situation, we will label the models in terms of the integer

$$n = k_1 + m_1 - 12 \quad (3.24)$$

where we assume without loss of generality that  $k_1 + m_1 \geq 12$ . We choose  $m_1 = m_2 = 3$  as before so that  $k_1 + k_2 = 18$ . We now set up the derivation of the spectrum implied by (3.20) upon maximal Higgsing of non-Abelian symmetries. The results of course depend on  $n$  or equivalently on the pair  $(k_1, k_2)$ . The strategy is to first implement breaking of the second  $E_6$  together with  $U(1)_D$  to  $G_0 \times U(1)_X$ , where  $U(1)_X$  is the appropriate ‘skew’ combination of  $U(1)_D$  and an  $E_6$  Cartan generator. Since  $k_1 \geq 9$ , the first  $E_6$  together with  $U(1)_X$  can then be broken to another ‘skew’  $U(1)_Y$ . The terminal gauge group is therefore  $G_0 \times U(1)_Y$  which by construction has a factorized anomaly polynomial. Except for  $n = 5$ , the terminal matter consists of  $G_0$  singlets charged under  $U(1)_Y$  plus a number of completely neutral hypermultiplets. The final step is to perform a toroidal compactification on  $T^2$  followed by transition to the Coulomb phase. This allows us to compare the resulting spectrum of vector and hypermultiplets with the Hodge numbers of candidate dual. The agreement between the spectra found for the different values of  $n$  and the corresponding Type-II compactifications is perfect. We refer the reader to [24] for further details.

One can also consider  $SU(2) \times U(1)^2$  backgrounds in each  $E_8$ . The  $U(1)$ ’s are embedded according to the branchings  $E_8 \supset SO(10) \times SU(4)$  and  $SU(4) \supset SU(2) \times SU(2)_A \times U(1)_B \supset SU(2) \times U(1)_A \times U(1)_B$ . The distribution of instanton numbers is chosen to be  $(k_1, m_{1A}, m_{1B}; k_2, m_{2A}, m_{2B}) = (k_1, 3, 2; k_2, 3, 2)$ , which can be shown to

guarantee a consistent spectrum. Notice that anomaly cancellation requires  $k_1 + k_2 = 14$  (in the absence of extra tensor multiplets from small instantons). The unbroken gauge group at the starting level is  $SO(10) \times U(1)^2 \times SO(10) \times U(1)^2$ . In this case the diagonal combinations  $Q_{AD} = (Q_{1A} + Q_{2A})$  and  $Q_{BD} = (Q_{1B} + Q_{2B})$  are anomaly-free whereas their orthogonal combinations are anomalous and are expected to be Higgsed away by a mechanism analogous to that explained before. Considering the different allowed values of  $k_i$  and  $m_{iA}, m_{iB}$  one again finds perfect agreement with the spectra found for the type C chains of Calabi-Yau compactifications. Equally satisfactory results are found for the type D chains.

To end this section, let us stress that the new elliptic fibrations (or the heterotic non-semisimple backgrounds) allow us to explore a new direction in moduli space, that of enhancing of abelian gauge groups. In the heterotic description it is evident that all the different families are connected through Higgsing of the  $U(1)$  factors. It has been shown in [24] that such a connection also exists between the CY spaces, and that it corresponds to conifold transitions. These are the first examples of extremal transitions of exactly the type considered in [47] (i.e. abelian) for which the heterotic version is known. Recently this observation has been employed to enlarge the class of B and C models using toric methods [54].

### 3.5 Exotic transitions and tensionless strings

As mentioned in section 2.2, a strong coupling problem is encountered in  $E_8 \times E_8$  compactifications. The gauge coupling constant of the group coming from the  $E_8$  with fewer instanton number diverges at the finite value of  $\phi$  satisfying

$$e^{-2\phi} = \frac{n}{2} \tag{3.25}$$

Due to the  $T$ -dualities with  $SO(32)$  vacua, the same behaviour is observed in these latter compactifications.

This phenomenon is also present in the F-theory framework, in which a geometrical interpretation can be given. The vev of the dilaton  $\phi$  is related to the ratio of the

Kähler classes  $k_b, k_f$  of the base and fiber  $\mathbb{P}_1$ 's in  $\mathbb{F}_n$ , through the equation [8]

$$e^{-2\phi} = \frac{k_b}{k_f} \quad (3.26)$$

One can determine that the size of the curve (divisor)  $w_1 = 0$  is  $k_b - n/2k_f$ , so that this cycle collapses for precisely the value (3.25). The structure of central charges of the SUSY algebra implies [4] that a BPS saturated string becomes tensionless at that point, and such object is certainly found in F-theory, as a Type-IIB 3-brane wrapped around the zero size cycle. The dynamics of these objects at the critical point is not infrared free, and the theory is at a non-trivial fixed point of the renormalization group. In some cases, this point lies at the intersection of different branches in moduli space, so that the tensionless string can mediate interesting phase transitions.

The physical behaviour at this point is encoded in the local geometry of the CY space near the vanishing curve. Actually, many features depend only on the geometry of the base, which once the curve has collapsed can be described as the projective space  $\mathbb{P}_2^{(1,1,n)}$ . Since it only depends on the value of  $n$ , the results are valid for the different A, B, C and D families. They also apply to  $SO(32)$  compactifications for the corresponding value of  $n$ .

For most values of  $n$ , not much is known about the theory at the critical point, since it is frozen at a Type-IIB strong coupling regime [55]. However the situation remains tractable for  $n = 0, 1, 2, 4$ , as we show in the following.

The case  $n = 0$  is special, since there is no gauge coupling divergence at finite  $\phi$ , and thus no singular behaviour. The case  $n = 2$ , although seemingly presents such a divergence, can avoid the singular point in moduli space by turning on generic vevs for hypermultiplets. These issues are related to heterotic/heterotic duality and will be further explored in section 4.

The cases  $n = 1, 4$  are interesting because the anomaly polynomial (2.6) becomes a perfect square, and can be cancelled by a GS mechanism using only the gravitational self-dual 2-form. Since the anti-selfdual 2-form in the tensor multiplet of the dilaton is not involved, a transition in which that tensor multiplet disappears is compatible with anomaly cancellation [4]. The new branch emerging from the transition point is parametrized by vevs for new hypermultiplets, and so is a Higgs branch. The geometric



version of this argument is that the base spaces  $\mathbb{F}_1, \mathbb{F}_4$  can be deformed to  $\mathbb{P}_2$ . Since  $h_{11}(\mathbb{P}_2) = 1$ , F-theory compactifications on elliptic fibrations over  $\mathbb{P}_2$  have no tensors and provide a description for the Higgs branch.

The deformation  $\mathbb{F}_1 \rightarrow \mathbb{P}_2$  is particularly simple to analyze, since  $\mathbb{P}_2^{(1,1,n)}$  is not singular for  $n = 1$  and the collapse of the curve  $w_1 = 0$  is simply a smooth blowing down <sup>5</sup>. Indeed, this simply amounts to setting  $w_1 = 1$  in the fibration equations of the type (3.9), yielding

$$y^2 = x^3 + \tilde{f}(z_1, z_2, w_2)x + \tilde{g}(z_1, z_2, w_2) \quad (3.27)$$

(and similar expressions for the B, C and D, or  $SO(32)$  cases). Since  $w_1$  no longer appears in the equations, new polynomial deformations are possible, reflecting the fact that the dilaton tensor multiplet has transformed into new hypermultiplets in order to preserve the cancellation of gravitational anomalies. Explicit counting of these deformations shows the Hodge numbers of the CY space change as

$$\begin{aligned} \Delta(h_{11}) &= -1 \\ \Delta(h_{12}) &= c_d - 1 \end{aligned} \quad (3.28)$$

where  $c_d$  is the Coxeter number of  $E_d$  and  $d = 8, 7, 6$  and  $5$  for models A, B, C and D. The groups  $E_d$  do enter in the heterotic picture as follows. Notice that for  $n = 1$ , complete Higgsing of the non-Abelian groups is possible in all models and this can be achieved by instantons of  $E_d \times U(1)^{8-d}$  that leave  $U(1)^{8-d}$  unbroken in each  $E_8$  (before further breaking to the diagonal combinations). In fact, the transition to  $n_T = 0$  occurs when  $k_2 \rightarrow k_2 + 1$ , where  $k_2$  corresponds to an  $E_d$  instanton. In the F-theory picture, the  $E_d$  groups appear because when the 2-cycle collapses in  $\mathbb{F}_1 \rightarrow \mathbb{P}_2$ , a 4-cycle of del Pezzo type shrinks in the CY [9], and there is a natural action of the Weyl group of  $E_d$  on the 2-cycles inside this complex surface. It can be checked that the del Pezzo surfaces obtained for the A, B, C and D fibrations are of the correct  $E_d$  type [24].

Existence of a Higgs branch with no tensor multiplets is also expected in the  $n = 4$

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<sup>5</sup>This process is locally identical to the shrinking of a small  $E_8$  instanton, mentioned in the transitions  $\mathbb{F}_n \rightarrow \mathbb{F}_{n\pm 1}$  in section 3.3.2

case. The Hodge numbers of the CY spaces change in the transition as

$$\begin{aligned}\Delta(h_{11}) &= -4 \\ \Delta(h_{12}) &= 1\end{aligned}\tag{3.29}$$

which can be understood as follows. As the tensor multiplet disappears, anomaly cancellation conditions force the appearance of 29 new hypermultiplets, 28 of which are employed in Higgsing the  $SO(8)$  gauge symmetry and one remains in the final spectrum, providing for the increase in  $h_{21}$ . Also 4 Cartan generators are lost, thus explaining the change in  $h_{11}$ . This process is actually the second transition shown in eq.(2.5), where it was shown to be compatible with anomaly cancellation. We stress that in all the  $n = 4$  compactifications we have mentioned there is a terminal  $SO(8)$ , so the transition to the Higgs branch is allowed in all cases (on the F-theory side, the existence of the corresponding  $D_4$  singularity is discussed in [9]).

## 4 Heterotic/Heterotic Duality

### 4.1 Self-Dual $D = 6$ , $N = 1$ Heterotic Vacua

Heterotic/heterotic duality in  $D = 6$ ,  $N = 1$  was first conjectured in refs. [10, 11, 12, 13] motivated by heterotic/five-brane duality [56] in  $D = 10$ . The duality would involve an equivalence between the strongly coupled heterotic compactified on K3 and the weakly coupled string obtained by wrapping four of the dimensions of the the five-brane on K3. The duality between the corresponding gravitational bosonic massless fields is given by the dictionary:

$$\phi \longleftrightarrow \tilde{\phi} = -\phi\tag{4.1}$$

$$G_{MN} \longleftrightarrow \tilde{G}_{MN} = e^{-\phi} G_{MN}\tag{4.2}$$

$$H \longleftrightarrow \tilde{H} = e^{-\phi} H\tag{4.3}$$

which shows that indeed this is a strong-weak coupling duality.

One of the most intuitive hints [12] for the existence of a  $D = 6$  heterotic/heterotic duality is the way in which the anomaly eight-form  $I_8$  factorizes into the product of

four-forms in  $D = 6$ . In fact,  $I_8 = X_4 \tilde{X}_4$ , with

$$\begin{aligned} X_4 &= \frac{1}{4(2\pi)^2} \left( \text{tr } R^2 - V_\alpha \text{tr } F_\alpha^2 \right) \\ \tilde{X}_4 &= \frac{1}{4(2\pi)^2} \left( \text{tr } R^2 - \tilde{V}_\alpha \text{tr } F_\alpha^2 \right) \quad , \end{aligned} \quad (4.4)$$

where  $\alpha$  runs over the gauge groups in the model. This very symmetric form of  $I_8$  suggests a duality under which one exchanges the tree-level Chern-Simons contribution to the Bianchi identity  $dH = \alpha'(2\pi)^2 X_4$  with the one-loop Green-Schwarz corrections to the field equations  $d\tilde{H} = \alpha'(2\pi)^2 \tilde{X}_4$ , in agreement with eq.(4.3). The kinetic term for the gauge bosons in eq.(2.9) also show a potential duality under the exchange  $\phi \leftrightarrow -\phi$  as long as  $\tilde{V}_\alpha = V_\alpha$ . As we discussed in chapter 1, in these expressions  $V_\alpha$  is a (positive) tree-level coefficient which is essentially the Kac-Moody level. On the other hand the coefficients  $\tilde{V}_\alpha$  are associated to the Green-Schwarz mechanism, depend on the massless spectrum of the model and they can be positive, negative or zero. It is thus clear that only certain  $D = 6$  compactifications, those for which  $\tilde{V}_\alpha = V_\alpha$  have a priori hope for presenting heterotic/heterotic duality.

In fact it is easy to obtain heterotic compactifications with  $\tilde{V}_\alpha = V_\alpha$ . In particular we already obtained such an example in chapter 2, eq.(2.19). It is obtained by a heterotic  $Spin(32)/Z_2$  compactification on K3 with a certain  $U(1)$  bundle corresponding to the  $U(1)$  decomposition  $SU(16) \times U(1) \in SO(32)$ . Using index theorems one finds, for instanton number 24, hypermultiplets transforming as  $2(\mathbf{120}) + 2(\overline{\mathbf{120}}) + 20(\mathbf{1})$ . Using the formulae we gave in chapter two one easily finds  $V_{SU(16)} = \tilde{V}_{SU(16)} = 2$ . Thus this provides the simplest example of heterotic/heterotic duality within the  $Spin(32)/Z_2$  theory. There are also perturbative  $E_8 \times E_8$  vacua which display heterotic/heterotic duality. Looking at eqs.(2.13), (2.14) one sees that  $E_8 \times E_8$  vacua with  $n = 2$  (which corresponds to instanton numbers  $(k_1, k_2) = (14, 10)$ ) yields duality for the gauge group in the first  $E_8$ . Then the second  $E_8$  is not self-dual but, as remarked in ref.[15], for generic regions in the hypermultiplet moduli space the second  $E_8$  is completely Higgsed away.

In fact the authors in refs.[10, 11, 12, 13] did not realized the existence of these *perturbative* realizations of heterotic/heterotic duality. Chronologically the first explicit realization of heterotic/heterotic duality in  $D = 6$ ,  $N = 1$  was the *non-perturbative*

duality of ref.[14] . These authors noticed that  $E_8 \times E_8$  compactifications with  $\tilde{V}_\alpha = 0$  present non-perturbative heterotic/heterotic duality. Since this possibility is obviously not symmetric under the exchange of  $V_\alpha$  and  $\tilde{V}_\alpha$ , it requires the dual gauge group to be generated by non-perturbative (small instanton) effects as suggested in [32] . We already mentioned in chapter 2 that vacua with  $n = 0$  are T-dual to  $Spin(32)/Z_2$  compactifications without vector structure and we know that such theories give rise to non-perturbative enhanced gauge groups when instantons collapse to zero size. This hypothesis is consistent with the fact that the gauge groups generated by these non-perturbative effects verify  $V_\alpha = 0$  (unlike the perturbative ones, which obviously have  $V_\alpha > 0$ ). The proposal can be justified [14] by considering this duality as arising from two (dual) ways of looking at the compactification of the  $D = 11$  M-theory on  $K3 \times S^1/Z_2$ . The two dual  $D = 6$  theories correspond to  $E_8 \times E_8$  heterotic  $n = 0$  compactifications on  $K3$ .

In fact both perturbative and non-perturbative realizations of heterotic/heterotic duality are two aspects of a single self-dual theory, as can be seen both in terms of Type-IIB orientifolds and F-theory.

## 4.2 Heterotic/Heterotic Duality and Type-IIB Orientifolds

Type-IIB ,  $Z_N$  orientifolds [57] are obtained by compactifying this string on  $T^4$  and further modding by  $\{\Omega, Z_N\}$  where  $\Omega$  is the worldsheet parity operation which acts like  $\Omega z = \bar{z}$  on the world-sheet complex coordinate [58, 59, 60, 17, 16].  $Z_N$  acts on  $T^4$  in the same way described in chapter 2 for heterotic orbifolds. The resulting vacua have  $N = 1$  SUSY in  $D = 6$ . The  $\Omega$ -twisted sectors are open strings, and consistency requires in general the presence of two types of boundaries: nine-branes and five-branes. Tadpole cancellation constraints the number of those as well as restricting the embedding of the  $Z_N$  symmetry on the nine-brane and five-brane Chan-Paton (CP) factors. We will be interested in the particular case of  $Z_2$  which corresponds to the Bianchi-Sagnotti-Gimon-Polchinski (BSGP) class of models [59, 60]. Notice that  $T^4/Z_2$  corresponds to an orbifold limit of K3 with 16  $Z_2$  fixed points. From the closed string sector of the orientifold one gets the usual gravity plus tensor multiplet. In addition,

one gets 4 moduli from the untwisted closed string and 16 more from the twisted ones. Tadpole cancellation requires the presence of 32 nine-branes (usual Type-I open string boundaries) and 32 (Dirichlet) five-branes. The model with maximal gauge symmetry is obtained for a configuration with no Wilson lines on the nine-branes and all five-branes sitting on the same fixed point. In this case the gauge group is  $U(16)_{99} \times U(16)_{55}$ , which corresponds to open strings stretching between a couple of nine(five)-branes respectively. Those strings also give rise to hypermultiplets transforming like  $2(\mathbf{120}, \mathbf{1})_{99} + 2(\mathbf{1}, \mathbf{120})_{55}$ . Open strings stretching between nine-branes and five-branes give rise to extra massless hypermultiplets transforming like  $(\mathbf{16}, \mathbf{16})_{59}$ . The reader may check that this massless spectrum indeed is anomaly free. In Type-I open strings, T-duality exchanges Neumann and Dirichlet boundary conditions and in the present case this means the exchange of five-branes and nine-branes. Indeed, this particular configuration is explicitly invariant under T-duality. The  $D = 6$  anomaly polynomial for this model was computed in ref.[23] and found to have the form:

$$A_8 = (R^2 - 2F_9^2)(R^2 - 2F_5^2). \quad (4.5)$$

Comparing this to eq.(2.7) one finds for  $U(16)_{99}$   $V_9 = 2$ ,  $\tilde{V}_9 = 0$  whereas for  $U(16)_{55}$  one has  $V_5 = 0$  and  $\tilde{V}_5 = 2$ . As remarked in [23], looked from the dual *heterotic* point of view one would say that the first  $U(16)$  is a perturbative gauge group with  $\tilde{V} = 0$ , similar to the  $n = 0$   $E_8 \times E_8$  vacua or the T-dual  $Spin(32)/Z_2$  models without vector structure. However, the second  $U(16)$  has non-perturbative origin as indicated by the fact that  $V = 0$  for that gauge group. From the Type-I point of view, T-duality exchanges both  $U(16)$ 's. Thus the non-perturbative heterotic/heterotic duality maps to usual T-duality on the Type-I dual side.

In this setting is easy to see the connection between the  $n = 2$  and  $n = 0$  realizations of heterotic/heterotic duality. Consider the above model with  $U(16)^2$  symmetry. If we give a non-vanishing vev to the hypermultiplets in  $(\mathbf{16}, \mathbf{16})_{59}$  the gauge group is broken to the diagonal one  $U(16)_{diag}$  and one has hypermultiplets transforming like  $4(\mathbf{120}) + 20(\mathbf{1})$ . But this is precisely the massless spectrum of the  $SO(32)$  perturbative model that we showed in the previous subsection. We also have  $V_{diag} = V_9 + V_5 = 2 + 0 = 2$  and  $\tilde{V}_{diag} = \tilde{V}_9 + \tilde{V}_5 = 0 + 2 = 2$  so that indeed  $\tilde{V}_{diag}/V_{diag} = 1$  as required to

have self-duality.

### 4.3 Heterotic/Heterotic Duality and F-theory

The F-theory description of heterotic/heterotic duality for  $n = 0$  turns out to be particularly simple. Since  $\mathbb{F}_0 = \mathbb{P}_1 \times \mathbb{P}_1$ , there is a natural symmetry in the moduli space which interchanges both  $\mathbb{P}_1$ 's. By eq.(3.26) this implies the change  $\phi \rightarrow -\phi$ , suggesting the identification of this action with heterotic/heterotic duality. Observe moreover that perturbative symmetries (associated to singular elliptic fibers over the 'base'  $\mathbb{P}_1$ ) are mapped to non-perturbative ones (corresponding to singular elliptic fibers over the 'fiber'  $\mathbb{P}_1$ ), thus explaining the non-trivial action of the duality on the hypermultiplets [14].

There is also an explanation for the existence of heterotic/heterotic duality for  $n = 2$ . As remarked above, the naive strong coupling singularity in the gauge factor with fewer instantons can be avoided by Higgsing that gauge symmetry, i.e. by turning on vevs for hypermultiplets. In the F-theory language this must be realized as a complex structure deformation of the fibration over  $\mathbb{F}_2$ , e.g.  $\mathbb{P}_4^{(1,1,2,8,12)}$  [24]. From the 243 deformations of this CY, 242 can be represented as polynomial deformations, and do not change the singular geometry on the base once the  $w_1 = 0$  curve has been blown down, i.e. once the  $\phi$ -value (3.25) has been reached. At that point the base becomes  $\mathbb{P}_2^{(1,1,2)}$ , with an  $A_1$  singularity. However, as remarked in [61, 8] there is a non-polynomial deformation. For a generic value of this, the CY cannot be embedded in the projective space  $\mathbb{P}_4^{(1,1,2,12,18)}$ . This in particular implies that when one reaches the problematic  $\phi$  value, the base space is not  $\mathbb{P}_2^{(1,1,2)}$ , but a deformation of it, with the  $A_1$  singularity properly smoothed. Then no tensionless string is found unless this modulus is tuned to zero.

Actually, this complex structure deformation transforms  $\mathbb{F}_2$  into  $\mathbb{F}_0$ , so that elliptic fibrations over both surfaces lead to identical CY's. Only when this parameter is turned off, i.e. at codimension one in the moduli space, precisely in the case that the CY can be embedded in  $\mathbb{P}_4^{1,1,2,12,18}$ , one meets the strong coupling singularity. The possibility of avoiding the singularity is also present in the physics of the non-critical string that

appear at that point [55]. Since the local geometry is hyperkähler, the string carries a tensor as well as an extra hypermultiplet degree of freedom. Its tension depends on all five parameters, and only vanishes when the scalar in the tensor multiplet *and* those in the hypermultiplet are properly tuned. For generic vevs for this hypermultiplet, the singularity is circumvented.

The results of section 4.2 concerning the equivalence of  $n = 0$  and  $n = 2$  compactifications in models with enhanced symmetries can also be reproduced in F-theory. As shown in ref.[52] one can construct  $n = 0$  compactifications with perturbative and non-perturbative enhanced gauge groups by forcing singular elliptic fibers over the ‘base’ and ‘fiber’  $\mathbb{P}_1$ ’s in  $\mathbb{F}_0$  (the particular configuration  $U(16) \times U(16)$  discussed in section 4.2 has been explicitly worked out in [30]). Upon a complex structure deformation, precisely the non-polynomial one mentioned above, the two curves merge smoothly at their intersection point to form a single curve of singular fibers, representing the diagonal subgroup of the original symmetry [52]. The final configuration is manifestly self-dual, and corresponds to a perturbative enhanced model from the  $n = 2$  viewpoint.

#### 4.4 $D = 4$ , $N = 2$ Heterotic/Heterotic Duality

Let us now consider the  $D = 4$  heterotic models obtained upon further compactification of the above  $D = 6$  heterotic duals on a 2-torus. This case was considered in ref. [13] and briefly mentioned in ref. [14]. The resulting  $N = 2$  theory has the usual toroidal vector multiplets  $S, T, U$  related to the coupling constant and the size and shape of the 2-torus. When the  $D = 6$  theory is dimensionally reduced to four dimensions, the underlying duality exchanges the roles of  $S$  and  $T$  [13, 62]. Including mirror symmetry on the torus, one thus expects complete  $S - T - U$  symmetry in this type of vacua [13, 14, 63, 42, 64, 65]. Thus, on top of the usual perturbative  $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$  dualities, a non-perturbative  $SL(2, \mathbb{Z})_S$   $S$ -duality [1] is expected. This  $N = 2$  model has the toroidal  $U(1)^4$  as generic gauge group, and as matter, 244 neutral hypermultiplets (it corresponds to the heterotic construction of model  $B$  of ref. [29]). At particular points in moduli space, enhanced gauge groups such as  $E_7 \times E_7$  can appear.

A natural question is the following: What is the  $D = 4$  equivalent of the  $\tilde{V}_\alpha = 0$  or  $\tilde{V}_\alpha = V_\alpha$  conditions we had in  $D = 6$  in order to have heterotic/heterotic duality? It turns out that the equivalent condition in  $D = 4$  can be phrased as a condition on the  $N = 2$   $\beta$ -functions of the gauge groups present at enhanced points in moduli space. Indeed, the  $N = 2$ ,  $D = 4$   $\beta$ -function of a given gauge factor can be written in terms of the corresponding  $D = 6$   $\tilde{V}_\alpha$  coefficient. More explicitly one finds [15]

$$\beta_\alpha^{N=2} = 12 \left( 1 + \frac{\tilde{V}_\alpha}{V_\alpha} \right) . \quad (4.6)$$

Thus, the condition to get heterotic/heterotic duality in  $N = 2$ ,  $D = 4$  models reads

$$\begin{aligned} \beta_\alpha^{N=2} &= 12 && \text{(symmetric } E_8 \times E_8 \text{ embeddings)} \\ \beta_\alpha^{N=2} &= 24 && \text{(non-symmetric } E_8 \times E_8 \text{ embeddings)} \end{aligned} . \quad (4.7)$$

Notice that in both cases the  $N = 2$  models are non-asymptotically free. In the first case ( $\beta_\alpha = 12$ ), consistently with the DMW hypothesis in  $D = 6$ , there should be points in moduli space in which new gauge groups of a non-perturbative origin should appear. Those are required to obtain full duality. In the second case ( $\beta_\alpha = 24$ ) this is not expected but explicit duality should be apparent.

One can think of the following consistency check . We know the form of the holomorphic  $N = 2$  gauge kinetic function  $f_\alpha$  for the gauge groups inherited from  $E_8 \times E_8$ . For a  $K3 \times T^2$  compactification of the type discussed here one has [66, 67]

$$f_\alpha = k_\alpha S_{inv} - \frac{\beta_\alpha^{N=2}}{4\pi} \log(\eta(T)\eta(U))^4 , \quad (4.8)$$

where  $\eta$  is the Dedekind function and  $S_{inv}$  is given by :

$$S_{inv} \equiv S - \frac{1}{2} \partial_T \partial_U h^{(1)}(T, U) - \frac{1}{2\pi} \log(J(T) - J(U)) + \text{const.} \quad (4.9)$$

Here  $h^{(1)}(T, U)$  is the moduli-dependent one-loop correction to the  $N = 2$  prepotential  $\mathcal{F}$  and  $J$  is the absolutely modular invariant function. Now, we know that the large- $T$  limit of  $f_\alpha$  must reproduce the result in eq. (2.9). It is not clear that this follows from the above equations. However one can check [15] that for large  $T$  one has  $S_{inv} \rightarrow S - T$  so that one gets for  $k_\alpha = 1$

$$\lim_{T \rightarrow \infty} f_\alpha = S + T \left( \frac{\beta_\alpha^{N=2}}{12} - 1 \right) = S + \frac{\tilde{V}_\alpha}{V_\alpha} T , \quad (4.10)$$



which is just the  $D = 4$  version of formula (2.9). We thus see that if any of the  $\beta_\alpha^{N=2}$  is smaller than 12, the large- $T$  limit gives rise to gauge kinetic terms of the wrong sign, which is just a four dimensional reflection of the  $D = 6$  singularity be discussed previously. Notice the different large- $T$  behaviour of the two heterotic/heterotic dualities under consideration. In the one proposed in [14] one has  $\tilde{V}_\alpha = 0$  and  $f \rightarrow S$ . In the alternative  $n = 2$  case [15] , one has  $f \rightarrow S + T$ , a  $S \leftrightarrow T$  invariant result.

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